Potentials $\vec{A}$ and Φ which satisfy the Lorenz condition $\Rightarrow$ "Lorenz gauge"

Under the Lorenz gauge;

- $$\vec{\nabla}^2 \Phi + \partial_t (\vec{\nabla} \cdot \vec{A}) = -\frac{\rho}{\varepsilon_0} \qquad (*)$$

$$\Rightarrow \vec{\nabla}^2 \Phi + \partial_t \left(-\frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = -\frac{\rho}{\varepsilon_0} \Rightarrow \underline{\underline{\vec{\nabla}^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}}}$$

- $$\vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \partial_t \Phi \right) = -\mu_0 J \qquad (**)$$

$$\Rightarrow \underline{\underline{\vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}}}$$

(speed of propagation c)

Wave Equations for Φ and $\vec{A}$!!
(follow from the Lorenz gauge choice)

Recall: also had the Coulomb gauge:

$\vec{\nabla} \cdot \vec{A} = 0$, Φ obeys:

$$(*) \quad \vec{\nabla}^2 \Phi = -\rho/\varepsilon_0 \Rightarrow \underbrace{\Phi(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|}}$$

this implies if $\rho(\vec{x}, t)$ changes in time, $\Phi(\vec{x}, t)$ also changes instantaneously !! (i.e. no dynamical wave equation)

$\Phi(\vec{x}, t)$ is instantaneous Coulomb potential

Now, with this choice of $\vec{\nabla} \cdot \vec{A} = 0$; $\vec{A}$ obeys:

$$(**) \quad \vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla} \Big(\underbrace{\vec{\nabla} \cdot \vec{A}}_{=0} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \Big) = -\mu_0 \vec{J}$$

$$\Rightarrow \vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \overbrace{\frac{1}{c^2} \vec{\nabla} \left(\frac{\partial \Phi}{\partial t} \right)}^{\text{note: has no curl}}$$

[Note: $\vec{\nabla} \times \vec{\nabla} \psi = 0 \quad \forall \psi$]

scalar function

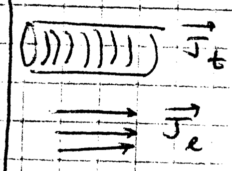
in principle, can calculate Φ from

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|}$$

so define: $\vec{J} = \vec{J}_\ell + \vec{J}_t$

longitudinal/irrotational current, $\vec{\nabla} \times \vec{J}_\ell = 0$

transverse, solenoidal current: $\underbrace{\vec{\nabla} \cdot \vec{J}_t = 0}$



Griffiths: Helmholtz Theorem: vector $\vec{F}$ uniquely specified by

$$\vec{F} = \underbrace{-\vec{\nabla} U}_{\vec{J}_\ell} + \underbrace{\vec{\nabla} \times \vec{W}}_{\vec{J}_t}$$

Goal: by writing $\vec{J}$ as such, see if can simplify the source terms: $-\mu_0 \vec{J} + \frac{1}{c^2} \vec{\nabla} (\partial_t \Phi)$

in general: vector identity

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{J}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{J}) - \vec{\nabla}^2\vec{J}$$

$$\Rightarrow \vec{\nabla}^2\vec{J} = \vec{\nabla}(\vec{\nabla}\cdot\vec{J}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{J})$$

or $$\vec{\nabla}^2 J_i = -\left[\left(-\vec{\nabla}(\vec{\nabla}\cdot\vec{J})\right)_i + \left(\vec{\nabla}\times(\vec{\nabla}\times\vec{J})\right)_i\right]$$ Poisson equation for J_i

We know how to solve this for J_i. Recall: $\vec{\nabla}^2\Phi(\vec{x}) = \frac{-\rho(\vec{x})}{\varepsilon_0}$ solution is:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$ [primed is integral over "sources"]

$\Rightarrow$ solution for the J_i is:

(see proof pp. 136b & 136c)

$$\Rightarrow \vec{J}(\vec{x}) = \underbrace{\frac{-1}{4\pi}\vec{\nabla}\int d^3x' \frac{\vec{\nabla}'\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}} + \underbrace{\frac{1}{4\pi}\vec{\nabla}\times\left(\vec{\nabla}\times\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)}$$

$\propto \vec{\nabla}$ (scalar function) $\qquad\qquad \propto \vec{\nabla}\times$ (vector)

$\underbrace{\vec{\nabla}\times\left(\vec{\nabla}(\text{scalar function})\right) = 0}_{\text{identify as } \vec{J}_\ell \text{ "longitudinal"}} \qquad \underbrace{\vec{\nabla}\cdot\left(\vec{\nabla}\times(\overrightarrow{\text{vector}})\right) = 0}_{\text{identify as } \vec{J}_t \text{ "transverse"}}$

Now, we have (continuity equation): $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$

$$\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$$

$$\Rightarrow \partial_t\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\partial_t\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|} = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{-\vec{\nabla}'\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

Take $\vec{\nabla}$ of both sides:

$$\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\right) = \frac{1}{\varepsilon_0}\left(\frac{1}{4\pi}\vec{\nabla}\int d^3x' \frac{-\vec{\nabla}'\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) = \frac{1}{\varepsilon_0}\vec{J}_\ell = c^2\mu_0\vec{J}_\ell$$

$$\Rightarrow \frac{1}{c^2}\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\right) = \mu_0\vec{J}_\ell$$ $\qquad c^2 = \frac{1}{\varepsilon_0\mu_0}$

Proof

$$\vec\nabla^2 J_i = -\Big[\underbrace{-\vec\nabla_i(\vec\nabla\cdot\vec J)}_{①} + \underbrace{(\vec\nabla\times(\vec\nabla\times\vec J))_i}_{②}\Big]$$

① solution is: $-\frac{1}{4\pi}\int d^3x' \frac{\vec\nabla'_i(\vec\nabla'\cdot\vec J)}{|\vec x-\vec x'|}$ $u=\frac{1}{|\vec x-\vec x'|}$

$\vec\nabla'\cdot\vec V = \vec\nabla'_i(\vec\nabla'\cdot\vec J)$
$= \vec\nabla'\cdot\underbrace{(\vec\nabla'\cdot\vec J)\hat x'_i}_{=\vec V}$

$$= -\frac{1}{4\pi}\oint_{S\to\infty} \frac{1}{|\vec x-\vec x'|}\underbrace{(\vec\nabla'\cdot\vec J)}_{0\text{ at }\infty}\hat x'_i\cdot\hat n'\,da + \frac{1}{4\pi}\int d^3x'\,\vec\nabla'\frac{1}{|\vec x-\vec x'|}\cdot(\vec\nabla'\cdot\vec J)\hat x'_i$$

$$= -\frac{1}{4\pi}\int d^3x'\,\vec\nabla_i\frac{1}{|\vec x-\vec x'|}(\vec\nabla'\cdot\vec J) = -\frac{1}{4\pi}\vec\nabla_i\int d^3x'\frac{1}{|\vec x-\vec x'|}(\vec\nabla'\cdot\vec J)$$

$$\Rightarrow \boxed{\vec J_\ell = -\frac{1}{4\pi}\vec\nabla\int d^3x'\frac{1}{|\vec x-\vec x'|}(\vec\nabla'\cdot\vec J)}\ \checkmark$$

[has no curl, so clearly is $\vec J_\ell$].

Now, to find $\vec J_t$, use: $\vec J=\vec J_t+\vec J_\ell \Rightarrow \vec J_t = \vec J-\vec J_\ell$ (follows from the Helmholtz Theorem)

$$\Rightarrow \vec J_t = \vec J + \frac{1}{4\pi}\vec\nabla\underbrace{\int d^3x'\frac{\vec\nabla'\cdot\vec J}{|\vec x-\vec x'|}}_{\text{work on this integral}}$$

$$\int d^3x'\frac{\vec\nabla'\cdot\vec J}{|\vec x-\vec x'|} = \underbrace{\int d^3x'\,\vec\nabla'\cdot\left(\frac{\vec J}{|\vec x-\vec x'|}\right)}_{\text{use divergence theorem}} - \int d^3x'\,\vec J\cdot\vec\nabla'\frac{1}{|\vec x-\vec x'|}$$

[by: $\vec\nabla\cdot(\psi\vec a)=\vec a\cdot\vec\nabla\psi+\psi\vec\nabla\cdot\vec a$]

$$= \underbrace{\oint_{S\text{ at }\infty} da'\frac{\vec J}{|\vec x-\vec x'|}\cdot\hat n'}_{0} - \int d^3x'\,\vec J\cdot\left(-\vec\nabla\frac{1}{|\vec x-\vec x'|}\right)$$

$$= \int d^3x'\,\vec J\cdot\vec\nabla\frac{1}{|\vec x-\vec x'|} = \vec\nabla\cdot\int d^3x'\frac{\vec J(\vec x')}{|\vec x-\vec x'|}$$

sources

[by $\vec\nabla\cdot(\psi\vec a)=\ldots$]

only place $\vec x$ appears

$$\Rightarrow \vec J_t = \vec J+\frac{1}{4\pi}\vec\nabla\left(\vec\nabla\cdot\int d^3x'\frac{\vec J(\vec x')}{|\vec x-\vec x'|}\right)$$

use: $\vec\nabla\times(\vec\nabla\times\vec a) = \vec\nabla(\vec\nabla\cdot\vec a)-\vec\nabla^2\vec a$, $\vec a$ = vector integral

Proof, con't

$$\Rightarrow \vec{J}_t = \vec{J} + \frac{1}{4\pi}\left[\vec{\nabla}\times\left(\vec{\nabla}\times\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) + \vec{\nabla}^2 \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right]$$

$$= \vec{J} + \frac{1}{4\pi}\left[\vec{\nabla}\times\left(\vec{\nabla}\times\int(\dots)\right) + \int d^3x' \, \vec{J}(\vec{x}') \underbrace{\vec{\nabla}^2 \frac{1}{|\vec{x}-\vec{x}'|}}_{= -4\pi\delta(\vec{x}-\vec{x}')}\right]$$

$$= \vec{J} + \frac{1}{4\pi}\left[\vec{\nabla}\times\left(\vec{\nabla}\times\int(\dots)\right) - 4\pi\vec{J}(\vec{x})\right]$$

$$= \frac{1}{4\pi}\vec{\nabla}\times\left(\vec{\nabla}\times\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) \checkmark$$

So our equation for $\vec{A}$ becomes:

$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \frac{1}{c^2}\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\right)$$

$$= -\mu_0\vec{J} + \mu_0\vec{J}_\ell = -\mu_0(\vec{J}-\vec{J}_\ell) = -\mu_0\vec{J}_t$$

$$\Rightarrow \quad \underline{\underline{\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}_t}} \quad (**)$$

(speed of propagation c)

"source" for the wave equation for $\vec{A}$ expressed entirely in terms of the transverse current

"Coulomb gauge" often called "transverse gauge"

Physical Notes:

① $\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$ implies Φ propagates instantaneously everywhere in space

but, $\vec{A}$ obeys wave equation ; more later

② $\vec{J}_t$ extends over all of space, even if $\vec{J}$ is localized

Green Functions for Wave Equation [Jackson, + Panofsky/Phillips, pp. 212-214] more physical/intuitive discussion in PP

So far, we have the following wave equations:

Lorenz Gauge:
$$\begin{cases} \vec{\nabla}^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \\ \vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \end{cases}$$

Coulomb Gauge:
$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}_t$$

all of the same general functional form: (in $(\vec{x},t)$)

$$\vec{\nabla}^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \overbrace{-4\pi f(\vec{x},t)}^{\text{source term}}$$

ψ: Φ, or the A_i components

To solve, introduce Fourier transforms to remove explicit time-dependence:

$$\psi(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(\vec{x},\omega)e^{-i\omega t}d\omega$$

$$f(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}f(\vec{x},\omega)e^{-i\omega t}d\omega$$

Inverse:
$$\psi(\vec{x},\omega) = \int_{-\infty}^{\infty}\psi(\vec{x},t)e^{i\omega t}dt$$

$$f(\vec{x},\omega) = \int_{-\infty}^{+\infty}f(\vec{x},t)e^{i\omega t}dt$$