Inserting Fourier transforms into wave equation gives:

$$\vec{\nabla}^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(\vec{x},t)$$

$$\vec{\nabla}^2\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\vec{x},\omega)e^{-i\omega t}d\omega\right] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\vec{x},\omega)e^{-i\omega t}d\omega\right] =$$

$$-4\pi\cdot\frac{1}{2\pi}\cdot\int_{-\infty}^{\infty}f(\vec{x},\omega)e^{-i\omega t}d\omega$$

Integrands:

$$\Rightarrow \vec{\nabla}^2\psi(\vec{x},\omega) - \frac{1}{c^2}(-i\omega)^2\psi(\vec{x},\omega) = -4\pi f(\vec{x},\omega)$$

$$\underbrace{(\vec{\nabla}^2 + k^2)\psi(\vec{x},\omega) = -4\pi f(\vec{x},\omega)}$$ for each ω value (in integrand)

Inhomogeneous Helmholtz wave equation, $k \equiv \frac{\omega}{c}$: wave number
elliptic partial differential equation; if $k=0$, reduces to Poisson equation

To solve the Helmholtz wave equation, find a Green function $G(\vec{x},\vec{x}')$ which satisfies the Helmholtz equation for a unit source at point $\vec{x}'$, and (then build up solution by superposition of point sources):

$$(\vec{\nabla}^2 + k^2)G_k(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$$. i.e., each unit point source satisfies this equation

If no bounding surfaces, solution must be spherically symmetric, depend only on $|\vec{x}-\vec{x}'|$, i.e., distance from source.

If spherically symmetric, (i.e., no θ, ϕ dependence) $\vec{\nabla}^2\psi \rightarrow \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$

If let $\vec{R} \equiv \vec{x}-\vec{x}'$ (i.e., radial vector from $\vec{x}'$), then Helmholtz equation becomes:

$$(\vec{\nabla}^2 + k^2)G_k(\vec{R}) = -4\pi\delta(\vec{R})$$

$$\frac{1}{R}\frac{d^2}{dR^2}(RG_k) + k^2G_k = -4\pi\delta(\vec{R})$$

Except at $\vec{R}=0$, G_k satisfies:

$$\frac{1}{R}\frac{d^2}{dR^2}(RG_k) + k^2G_k = 0 \Rightarrow \frac{d^2}{dR^2}(RG_k) + k^2(G_kR) = 0$$

[Solution to: $\frac{d^2}{dx^2}u(x) + k^2u(x) = 0$ is $u(x) = (\text{constant})e^{\pm ikx}$]

$$\rightarrow RG_k(R) \propto e^{\pm ikR} \Rightarrow RG_k(R) = Ae^{+ikR} + Be^{-ikR}$$

To find A, B: Consider:

In the neighborhood of the singular point, $\vec{R}=0$, ($k=\frac{\omega}{c}$ finite)

$$G_k(R) = \frac{A}{R}e^{+ikR} + \frac{B}{R}e^{-ikR} \xrightarrow[R\to 0]{} \frac{A}{R} + \frac{B}{R} = \frac{A+B}{R}$$

In this limit, then we get:

$$\left(\vec{\nabla}^2 + k^2\right)\frac{A+B}{R} = -4\pi\,\delta(\vec{R})$$

Integrate "around" $\vec{R}=0$: (e.g., tiny sphere)

$$\int d^3x \left(\vec{\nabla}^2 + k^2\right)\frac{A+B}{R} = -4\pi\,\delta(\vec{R}) \quad , \quad \vec{\nabla}^2\left(\frac{1}{R}\right) = \vec{\nabla}^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

$$\int_{V:\vec{R}=0} d^3x\left[-4\pi\,\delta(\vec{R})\,(A+B)\right] + \underbrace{\int_{V:\vec{R}=0} \overbrace{d^3x}^{(r^2dr)\sin\theta d\theta d\varphi}\, k^2\,\frac{(A+B)}{R}}_{\to 0 \text{ as } \vec{R}\to 0} = -\int_{V:\vec{R}=0} 4\pi\,\delta(\vec{R})\,d^3x$$

$$\Rightarrow A + B = 1$$

just like in QM!

$$G_k(\vec{R}) = \underbrace{\frac{A}{R}e^{ikR}}_{\substack{\text{spherical wave}\\ \text{propagating (diverging)}\\ \text{from origin}}} + \underbrace{\frac{B}{R}e^{-ikR}}_{\substack{\text{spherical wave converging}\\ \text{towards origin}}} \; ; \; A+B=1$$

Physical basis of A and B: from boundary conditions in time. e.g., if a "source" turns on at $t=0$ (with nothing before), $A=1$, $B=0$. (at $\vec{R}=0$)

(waves radiated from $\vec{R}=0$)

Now, let: $G_k^{(+)}(R) \equiv \dfrac{e^{ikR}}{R}$ $\quad G_k^{(-)}(R) \equiv \dfrac{e^{-ikR}}{R}$ — these give the spatial dependence of the Green functions!

$$\Rightarrow \left[G_k(R) = A\,G_k^{(+)} + B\,G_k^{(-)}\right]$$

Still need to construct the time-dependence of the Green functions!

More generally, the Green function must satisfy for a point source (in position AND time)

$$\vec{\nabla}^2 G^{(\pm)}(\underbrace{\vec{x},t}_{\text{obs. pt.}};\underbrace{\vec{x}',t'}_{\text{source}}) - \frac{1}{c^2}\frac{\partial^2 G^{(\pm)}}{\partial t^2} = \underbrace{-4\pi\,\delta(\vec{x}-\vec{x}')\,\delta(t-t')}_{\text{source term for point source}} \quad \left[= -4\pi \underbrace{f(\vec{x},t)}_{\text{source function}}\right]$$

(form of original wave equation) before transforming to ω-space

Thus, we can read off from this that we must have:

$$-4\pi f(\vec{x},t) = -4\pi\,\delta(\vec{x}-\vec{x}')\,\delta(t-t') = -4\pi\,\underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\vec{x},\omega)\,e^{-i\omega t}\,d\omega}_{\text{Fourier transform of } f(\vec{x},t)}$$

Satisfied if:

$$f(\vec{x},\omega) = \delta(\vec{x}-\vec{x}')\,e^{i\omega t'} \quad \text{as then:}$$

$$-4\pi\cdot\frac{1}{2\pi}\int_{-\infty}^{+\infty}\delta(\vec{x}-\vec{x}')\,e^{i\omega(t'-t)}\,d\omega = -4\pi\,\delta(\vec{x}-\vec{x}')\cdot\underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega(t'-t)}\,d\omega}_{=\,\delta(t-t')} \checkmark$$

Thus, the source term in the Helmholtz wave equation in ω-space becomes

$$(\vec{\nabla}^2 + k^2)\,G_k(\vec{R},\omega) = -4\pi f(\vec{x},\omega) = -4\pi\,\delta(\vec{x}-\vec{x}')\,e^{i\omega t'}$$

with: $G_k = G^{(\pm)}(R)\,e^{i\omega t'}$ as:

$$(\vec{\nabla}^2 + k^2)\,G^{(\pm)}(R)\,e^{i\omega t'} = -4\pi\,\delta(\vec{x}-\vec{x}')\,e^{i\omega t'} \checkmark$$

~~Thus, finally, the time dependent Green function becomes: (in real $\vec{x}, t$ space)~~

$$\left[\psi(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(\vec{x},\omega)\,e^{-i\omega t}\,d\omega\right]$$

$$\Rightarrow\quad G^{(\pm)}(R,t,t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G^{(\pm)}(R)\,e^{i\omega t'}\,e^{-i\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{\pm ikR}}{R}\,e^{-i\omega(t-t')}\,d\omega$$

For: $k = \omega/c$, $\tau \equiv t - t'$:

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{\pm i\frac{\omega}{c}R}}{R}\,e^{-i\omega(t-t')}\,d\omega = \frac{1}{2\pi R}\int_{-\infty}^{\infty} d\omega\,\exp\left[-i\omega\tau \pm i\frac{\omega R}{c}\right]$$

$$= \frac{1}{2\pi R}\int_{-\infty}^{+\infty} d\omega\,\exp\left[-i\omega\left(\tau \mp \frac{R}{c}\right)\right] = \frac{1}{R}\cdot\underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\,\exp\left[i\omega\left(-\left(\tau \mp \frac{R}{c}\right)\right)\right]}$$

$$= \frac{1}{R}\cdot\delta\left(-\left(\tau \mp \frac{R}{c}\right)\right) \qquad \text{recall: } \delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\,e^{iu(x-x')}$$

$$\Rightarrow\quad G^{(\pm)}(\vec{x},t;\vec{x}',t') = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left(-\left((t-t') \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)\right)$$

$$= \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left(t' - \left(t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)\right)$$