Physical Interpretation:

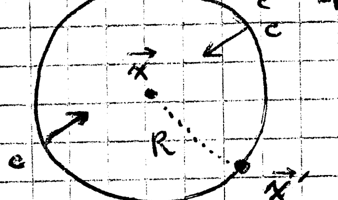
① $G^{(+)} = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left(t' - \left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right)$ is the "retarded" Green function,

Causal behavior! Effect perceived at point $\vec{x}$ at time t due to source at point $\vec{x}'$ at time $t' = t - \frac{R}{c} \Rightarrow t = t' + \frac{R}{c}$

source acts at an earlier, or "retarded" time $(t' < t)$

Visualization:

Observer at point $\vec{x}$, some source at point $\vec{x}'$:



← Sphere, centered on $\vec{x}$. Contracts with radial velocity c, in the process "collecting information" about sources, such that converges at $\vec{x}$ at time t. The time at which this "information-collecting sphere" passes (encloses) the source at $\vec{x}'$ is at time $t' = t - \frac{R}{c}$, such that the effect is "felt" at $\vec{x}$ at time t.

② $G^{(-)} = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left(t' - \left(t + \frac{|\vec{x}-\vec{x}'|}{c}\right)\right)$ is the "advanced" Green function.

Violate causality! Effect at $\vec{x}$ at time t due to source at point $\vec{x}'$ at time $t' = t + \frac{R}{c}$ i.e., in the future! (effect precedes the cause). Theoretical interest.

$(t' > t!)$

Back to our wave equation: $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(\vec{x}, t) \qquad \psi = \Phi, A_i$

We found the Green function that satisfies this wave equation for a point source, $G^{(\pm)}(\vec{x}, t; \vec{x}', t')$. Thus, for an extended source (in position and/or time), the solution to the wave equation is simply:

$$\psi^{(\pm)}(\vec{x}, t) = \int dt' \int d^3x' \underbrace{G^{(\pm)}(\vec{x}, t; \vec{x}', t')}_{\text{solution for point source}\ \delta(\vec{x}-\vec{x}')\,\delta(t-t')} \overbrace{f(\vec{x}', t')}^{\text{source distribution}}$$

Write this explicitly:

Suppose exists source $f(\vec{x}', t')$ localized in time and space. (using G^{+})

$$\psi(\vec{x}, t) = \int dt' \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left(t' - \left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right) f(\vec{x}', t')$$

$$= \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \underbrace{\left[f(\vec{x}', t') \right]_{ret}}$$

Jackson notation $[\ldots]_{ret}$ means t' is to be evaluated at the retarded time $t' = t - \frac{|\vec{x}-\vec{x}'|}{c}$, as enforced by δ-function in $\int dt'$.

Retarded Solutions for the Fields

Now we can solve:

$$\left\{ \begin{aligned} \vec{\nabla}^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \quad \left(= -4\pi f(\vec{x}, t)\right) \\ \vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} \quad \left(= -4\pi \vec{f}(\vec{x}, t)\right) \end{aligned} \right\} \text{Lorenz gauge}$$

Solutions are in Lorenz Gauge:

$$\Phi(\vec{x}, t) = \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \cdot \frac{[\rho(\vec{x}', t')]_{ret}}{4\pi\varepsilon_0}$$

$$= \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{1}{R} [\rho(\vec{x}', t')]_{ret} \qquad \vec{R} \equiv \vec{x} - \vec{x}'$$

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} [\vec{J}(\vec{x}', t')]_{ret} \qquad R = |\vec{x}-\vec{x}'|$$

(starting point for Ch. 9 on Radiation) $\qquad \hat{R} = \frac{\vec{R}}{R}$

In principle, $\vec{E}$, $\vec{B}$ can be calculated from these. EXAMPLE 142b-c

Often useful: retarded integral solutions in terms of sources for the fields.

From the Maxwell Equations: (in vacuum)

$$\left. \vec{\nabla} \times \vec{E} = -\partial_t \vec{B} \right\} \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = \vec{\nabla}\left(\frac{\rho}{\varepsilon_0}\right) - \vec{\nabla}^2 \vec{E}$$

$$\vec{\nabla} \times (-\partial_t \vec{B}) = -\partial_t \vec{\nabla} \times \vec{B} = -\partial_t \left(\mu_0 \vec{J} + \mu_0 \partial_t \vec{D}\right)$$

$$= -\mu_0 \partial_t \vec{J} - \mu_0 \partial_t^2 \vec{D} = -\mu_0 \partial_t \vec{J} - \mu_0 \varepsilon_0 \partial_t^2 \vec{E}$$

$$\Rightarrow \vec{\nabla}\left(\frac{\rho}{\varepsilon_0}\right) - \vec{\nabla}^2 \vec{E} = -\mu_0 \partial_t \vec{J} - \frac{1}{c^2} \partial_t^2 \vec{E}$$

$$\vec{\nabla}^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{\varepsilon_0} \left(-\vec{\nabla}\rho - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t}\right) \quad // \qquad \left[= -4\pi f(\vec{x}, t)\right]$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \partial_t \vec{D}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \frac{\vec{B}}{\mu_0}\right) = \vec{\nabla} \times \vec{J} + \partial_t \vec{\nabla} \times \vec{D}$$

$$\vec{\nabla}\underbrace{(\vec{\nabla} \cdot \vec{B})}_{=0} - \vec{\nabla}^2 \vec{B} = \mu_0 \vec{\nabla} \times \vec{J} + \mu_0 \partial_t \vec{\nabla} \times \varepsilon_0 \vec{E} = \mu_0 \vec{\nabla} \times \vec{J} + \frac{1}{c^2} \partial_t \left(-\partial_t \vec{B}\right)$$

$$\Rightarrow \vec{\nabla}^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = -\mu_0 \vec{\nabla} \times \vec{J} \quad // \qquad \left[= -4\pi f(\vec{x}, t)\right]$$

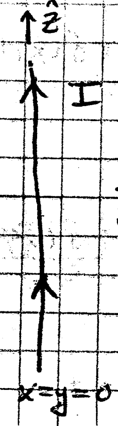
Example:

Infinite straight wire carries current

$$I(t) = \begin{cases} 0 & t \le 0 \\ I & t > 0 \end{cases}$$

(assume "on" instantaneously everywhere at $t = 0^+$)

"artificial"



$$\vec{J}(t) = \underbrace{I(t)\,\delta(x)\,\delta(y)}_{\text{correct dimensions!}}\,\hat{z} \qquad t>0$$

$\rho = 0$ (wire electrically neutral) $\Rightarrow \Phi = 0$

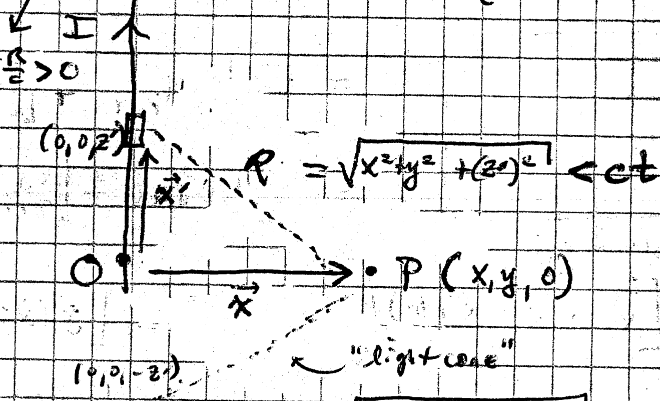
Find $\vec{E}$ and $\vec{B}$.

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{R}\left[\vec{J}(\vec{x}',t')\right]_{ret}$$

$$= \frac{\mu_0}{4\pi}\int dz'\int dy'\int dx'\,\frac{1}{R}\left[I(t')\,\delta(x')\,\delta(y')\right]_{ret}\hat{z}\,; \quad \frac{1}{R} = \frac{1}{|\vec{x}-\vec{x}'|}$$

$$= \frac{\mu_0}{4\pi}\int dz'\,\frac{1}{|\vec{x}-\vec{x}'|}\,I\,\hat{z} \qquad t' = t - \frac{R}{c}$$

for $t' = t - \frac{R}{c} > 0$



Thus, "source" points satisfying $R < ct$ contribute to the integral!

if $R < ct \Rightarrow$

$$\sqrt{x^2+y^2+(z')^2} < ct$$

$$\Rightarrow (z')^2 < c^2t^2 - (x^2+y^2) \Rightarrow |z'| < \sqrt{c^2t^2-(x^2+y^2)}$$

$$\Rightarrow \vec{A}(\vec{x},t) = \frac{\mu_0 I}{4\pi}\,2\int_0^{\sqrt{c^2t^2-(x^2+y^2)}} dz'\,\frac{1}{\left[x^2+y^2+(z')^2\right]^{1/2}}\,\hat{z}$$

$$\left[\text{using: } \int \frac{du}{(a^2+u^2)^{1/2}} = \ln\left(u + \sqrt{u^2+a^2}\right)\right]$$

$$\vec{A}(\vec{x},t) = \frac{\mu_0 I}{2\pi}\left[\ln\left(\sqrt{c^2t^2-(x^2+y^2)} + \sqrt{c^2t^2}\right) - \ln\left(\sqrt{x^2+y^2}\right)\right]\hat{z}$$

$$= \frac{\mu_0 I}{2\pi}\cdot\ln\left(\frac{ct + \sqrt{c^2t^2-(x^2+y^2)}}{\sqrt{x^2+y^2}}\right)\hat{z}$$

for $ct \ge x^2+y^2$

if $ct = \sqrt{x^2+y^2}$

$\vec{A} = 0$ ✓

$$\vec{E} = -\partial_t \vec{A}$$

$$= -\frac{\mu_0 I}{2\pi} \frac{\cancel{\sqrt{x^2+y^2}}}{ct+\sqrt{c^2t^2-(x^2+y^2)}} \cdot \left(c + \frac{1}{\cancel{2}}(c^2t^2-(x^2+y^2))^{-1/2} \cdot \cancel{2}c^2t\right) \cdot \frac{1}{\cancel{(x^2+y^2)^{1/2}}} \hat{z}$$

$$= -\frac{\mu_0 I}{2\pi} \frac{1}{ct+\sqrt{c^2t^2-(x^2+y^2)}} \left[c + \frac{c^2 t}{\sqrt{c^2t^2-(x^2+y^2)}}\right] \hat{z}$$

$$= -\frac{\mu_0 I c}{2\pi} \left[\frac{1}{ct+\sqrt{c^2t^2-(x^2+y^2)}}\right] \left[\frac{\sqrt{c^2t^2-(x^2+y^2)}+ct}{\sqrt{c^2t^2-(x^2+y^2)}}\right] \hat{z}$$

$$= -\frac{\mu_0 I c}{2\pi} \frac{1}{\sqrt{c^2t^2-(x^2+y^2)}} \hat{z}$$

[The time-dependent $\vec{A} \to \vec{B}$ gives rise to $\vec{E}(t)$!]

As $t \to \infty$, $\vec{E} \to 0$ (as $\rho = 0$)

$\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{A} = A_z \hat{z} = A_z(x,y,t)\hat{z}$

in cylindrical coordinates: (ρ, ϕ, z)

$$\vec{B} = \hat{\phi}\left(-\frac{\partial A_z}{\partial \rho}\right), \quad \rho = \sqrt{x^2+y^2}$$

$$= -\hat{\phi}\left[\frac{\mu_0 I}{2\pi} \frac{\rho}{ct+\sqrt{c^2t^2-\rho^2}}\right] \cdot \left[\frac{\rho\left(\frac{1}{\cancel{2}}\right)(c^2t^2-\rho^2)^{-1/2} \cdot (\cancel{2}\rho) - \left(ct+\sqrt{c^2t^2-\rho^2}\right)}{\rho^2}\right]$$

$$= -\hat{\phi}\left[\frac{\mu_0 I}{2\pi} \frac{\rho}{ct+\sqrt{c^2t^2-\rho^2}}\right] \cdot \left[\frac{-1}{\sqrt{c^2t^2-\rho^2}} - \frac{ct+\sqrt{c^2t^2-\rho^2}}{\rho^2}\right]$$

$$= -\hat{\phi}\left[\frac{\mu_0 I}{2\pi} \frac{\rho}{ct+\sqrt{c^2t^2-\rho^2}}\right] \cdot \frac{-\cancel{\rho^2} - ct\sqrt{c^2t^2-\rho^2} - c^2t^2 + \cancel{\rho^2}}{\sqrt{c^2t^2-\rho^2}\,\rho^2}$$

$$= +\hat{\phi} \frac{\mu_0 I}{2\pi} \frac{\rho c t}{\rho^2\sqrt{c^2t^2-\rho^2}}$$

$$= \frac{\mu_0 I}{2\pi} \frac{1}{\rho} \frac{ct}{\sqrt{c^2t^2-\rho^2}} \hat{\phi}$$ ✓

Note: As $t \to \infty$, $\vec{B} \to \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$ ✓ (usual result for infinite wire!)

Poynting's Theorem

Consider charge q in external electromagnetic fields, $\vec{E}$ and $\vec{B}$ (static). Work done by fields is:

$$dW = \vec{F}\cdot d\vec{r} = (q\vec{E} + q\vec{v}\times\vec{B})\cdot d\vec{r}$$

$\Rightarrow$ instantaneous rate of doing work is:

$$\frac{dW}{dt} = (q\vec{E} + q\vec{v}\times\vec{B})\cdot\vec{v} = q\vec{E}\cdot\vec{v} + q(\vec{v}\times\vec{B})\cdot\vec{v} = q\vec{E}\cdot\vec{v}$$ as $\vec{B}$ does no work!

$\Rightarrow$ total rate of doing work over a continuous distribution of charge and fields is: $\underbrace{\int_V d^3x\, \vec{J}\cdot\vec{E}}$ (power) $[\vec{J} = nq\vec{v}] = \rho\vec{v}$ $\quad$ n: #/m^3 $\quad$ $[dq = \rho\, d^3x]$

Represents rate of conversion electromagnetic energy $\rightarrow$ mechanical energy (or heat/thermal energy) of charge distribution.

Energy is conserved!! So this must be balanced by a corresponding rate of decrease of energy in the electromagnetic field in the volume V.

So, we see: $\quad [\vec{\nabla}\times\vec{H} = \vec{J} + \partial_t\vec{D}]$

$$\int_V d^3x\, \vec{J}\cdot\vec{E} = \int_V d^3x \left[(\vec{\nabla}\times\vec{H} - \partial_t\vec{D})\cdot\vec{E}\right]$$

$$= \int_V d^3x \left[\vec{E}\cdot(\vec{\nabla}\times\vec{H}) - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right]$$

$$\left[\text{use: } \vec{\nabla}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{\nabla}\times\vec{a}) - \vec{a}\cdot(\vec{\nabla}\times\vec{b})\right] \quad \vec{a} = \vec{E},\ \vec{b} = \vec{H}$$

$$= \int_V d^3x \left[-\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot\partial_t\vec{D}\right] \qquad \vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$$

$$= -\int_V d^3x \left[\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right]$$

[Assume: ① $\vec{D} = \varepsilon\vec{E}$ ② $\vec{B} = \mu\vec{H}$ (linearity)
③ valid for time-varying fields]

$$= -\int_V d^3x \left[+\frac{1}{2}\frac{\partial}{\partial t}(\vec{H}\cdot\vec{B}) + \frac{1}{2}\frac{\partial}{\partial t}(\vec{D}\cdot\vec{E}) + \vec{\nabla}\cdot(\vec{E}\times\vec{H})\right]$$

$$\Rightarrow \underbrace{-\int_V \vec{J}\cdot\vec{E}\, d^3x}_{} = +\int_V d^3x \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot(\vec{E}\times\vec{H})\right]$$

rate of <u>decrease</u> of energy in the electromagnetic field in volume V = rate of increase of mechanical energy

$u \equiv \frac{1}{2}(\vec{H}\cdot\vec{B} + \vec{D}\cdot\vec{E})$: total energy density