work on: $$\vec{\nabla}^2 \frac{1}{(r^2+a^2)^{1/2}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \partial_r \frac{1}{(r^2+a^2)^{1/2}}\right)$$

in spherical coordinates (see back cover of book)

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot -\frac{1}{2}(r^2+a^2)^{-3/2}\cdot 2r\right)$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{r}{(r^2+a^2)^{3/2}}\right)$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^3}{(r^2+a^2)^{3/2}}\right) \longleftarrow \text{see other page}$$

$$= -\frac{1}{r^2}\left[\frac{3r^2a^2}{(r^2+a^2)^{5/2}}\right] = -\frac{3a^2}{(r^2+a^2)^{5/2}}$$

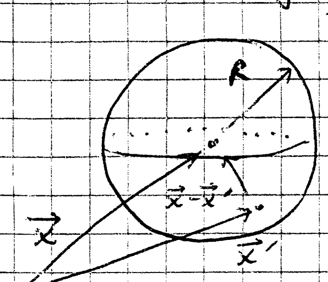
Using!

$$\Rightarrow \vec{\nabla}^2 \frac{1}{(r^2+a^2)^{1/2}} = -\frac{3a^2}{(r^2+a^2)^{5/2}}$$

$$\Rightarrow \vec{\nabla}^2 \Phi(\vec{x}) = \lim_{a\to 0}\left[\frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}\,')\left(\frac{-3a^2}{(r^2+a^2)^{5/2}}\right) d^3x'\right]$$

This is well behaved everywhere in space, except for $r=0$ in the $\lim_{a\to 0}$!

To evaluate the integral, define a sphere of radius R $(\gg a)$ to be centered at $\vec{x}$:



choose R such that ρ changes little over the volume of the sphere (i.e., $\rho \cong$ constant over the sphere volume)

the contributions to integral from $r>R$ vanish as $\frac{a^2}{r^2} \ll 1$

Note: $$\frac{1}{(r^2+a^2)^{5/2}} = \frac{1}{(r^2)^{5/2}\left(1+\frac{a^2}{r^2}\right)^{5/2}} \approx \frac{1}{r^5}\left(1 + \frac{5}{2}\frac{a^2}{r^2} + \dots\right)$$

So do the integral over the sphere: (in spherical coordinates: $d^3x' = r^2\sin\theta\, dr\, d\theta\, d\phi$)

$$\vec{\nabla}^2 \Phi(\vec{x}) \simeq \lim_{a\to 0}\left[-\frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\, d\theta \int_0^R (dr\, r^2)\, \rho(\vec{x}\,')\cdot\left(\frac{3a^2}{(r^2+a^2)^{5/2}}\right)\right]$$

Now, let's expand $\rho(\vec{x}\,') = \rho(\vec{x}) + \dots$

$$= \lim_{a\to 0}\left[-\frac{1}{\varepsilon_0}\int_0^R (r^2\, dr)\left[\rho(\vec{x}) + \dots\right]\frac{3a^2}{(r^2+a^2)^{5/2}}\right]$$

$$\cong \lim_{a\to 0}\left[-\frac{1}{\varepsilon_0}\,\rho(\vec{x})\int_0^R \frac{3a^2 r^2\,dr}{(r^2+a^2)^{5/2}}\right]$$

use: (ultimately) $\displaystyle\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2(a^2+u^2)^{1/2}}$

$$= \lim_{a\to 0}\left[-\frac{\rho(\vec{x})}{\varepsilon_0}\left(\cancel{3}a^2 r\cdot\left(-\frac{1}{\cancel{3}}\right)(r^2+a^2)^{-3/2} + \cancel{3}a^2\left(\frac{1}{\cancel{3}}\right)\frac{r}{(a^2+r^2)^{1/2}\,a^2}\right)\Bigg|_{r=0}^{r=R}\right]$$

$$= -\frac{\rho(\vec{x})}{\varepsilon_0}\cdot\lim_{a\to 0}\left[\underbrace{\frac{-a^2R}{(R^2+a^2)^{3/2}} + \frac{R}{(a^2+R^2)^{1/2}}}\right]$$

$$= -\frac{\rho(\vec{x})}{\varepsilon_0}\cdot\left(1 + O\left(\frac{a^2}{R^2}\right)\right) \longrightarrow -\frac{\rho(\vec{x})}{\varepsilon_0} \checkmark$$

Mathematical curiosity that follows ...
From this exercise, we see that we can identify:

$$\vec{\nabla}^2\Phi(\vec{x}) = \lim_{a\to 0}\left[\frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}')\,\underbrace{\vec{\nabla}^2\frac{1}{(r^2+a^2)^{1/2}}}\,d^3x'\right]$$

We have that:

$$\lim_{a\to 0}\vec{\nabla}^2\frac{1}{(r^2+a^2)^{1/2}} = \lim_{a\to 0}\left[-\frac{3a^2}{(r^2+a^2)^{1/2}}\right] = \begin{cases} 0 & \text{for } r\neq 0\\ ? & r = 0\end{cases}$$

but integral over all of space: $\displaystyle\lim_{a\to 0}\underbrace{\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta}_{=4\pi}\int_0^{R\to\infty}\underbrace{\vec{\nabla}^2\frac{1}{(r^2+a^2)^{1/2}}\,r^2dr}_{=-1} = 4\pi$

$= -1$ as we just showed

$\displaystyle\lim_{a\to 0}\int_0^R \vec{\nabla}^2\left(\frac{1}{(r^2+a^2)^{1/2}}\right)r^2\,dr = -1+O\left(\frac{a^2}{R^2}\right)$

$= -4\pi$

implies we can define:

$$\vec{\nabla}^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

[Recall: $\vec{r} = \vec{x}-\vec{x}'$ as defined earlier]

Green's Theorem:

If electrostatics was limited to discrete/continuous charge distributions with no boundary surfaces, could trivially use

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}, \quad \text{and } \vec{E} = -\vec{\nabla}\Phi$$

to solve any problem! wouldn't need Poisson/Laplace equations!

However, most practical/realistic problems involve finite regions of space, w/ or w/o charge inside, and prescribed boundary conditions on the surfaces.

Need new mathematical tools to handle the boundary conditions:

Start with Gauss's Theorem (divergence theorem):

$$\int_V \vec{\nabla}\cdot\vec{A}\, d^3x = \oint_{\substack{S \\ (\text{closed})}} \vec{A}\cdot\hat{n}\, da$$

$\vec{A}$: well-behaved vector field defined in V bounded by closed surface S

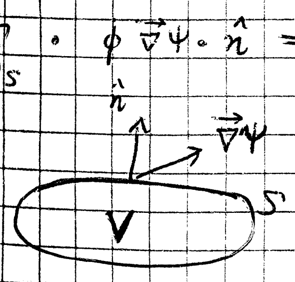
Let: $\vec{A} = \phi\,\vec{\nabla}\psi$, $\quad \phi, \psi$: arbitrary scalar fields

Note: [using the vector identity $\vec{\nabla}\cdot(f\vec{V}) = \vec{V}\cdot\vec{\nabla}f + f\,\vec{\nabla}\cdot\vec{V}$]

in $\int_V$ • $\vec{\nabla}\cdot(\underbrace{\phi}_{"f"}\,\underbrace{\vec{\nabla}\psi}_{"\vec{V}"}) = \vec{\nabla}\psi\cdot\vec{\nabla}\phi + \phi\,\vec{\nabla}\cdot(\vec{\nabla}\psi)$

$$= \vec{\nabla}\psi\cdot\vec{\nabla}\phi + \phi\,\vec{\nabla}^2\psi$$

in $\oint_S$ • $\phi\,\vec{\nabla}\psi\cdot\hat{n} = \phi\cdot(\text{projection of } \vec{\nabla}\psi \text{ onto } \hat{n}) \equiv \phi\,\underbrace{\frac{\partial\psi}{\partial n}}$



in coordinate system with $\hat{n}$ as one of the coordinates... → derivative of ψ w.r.t. $\hat{n}$ (component of $\vec{\nabla}\psi$ along $\hat{n}$) $(\vec{\nabla}\psi\cdot\hat{n})$

Plug into the divergence theorem: Green's First Identity

$$\boxed{\int_V \left[\vec{\nabla}\psi\cdot\vec{\nabla}\phi + \phi\,\vec{\nabla}^2\psi\right] d^3x = \oint_S \phi\,\frac{\partial\psi}{\partial n}\, da} \qquad (*)$$

Now let's write $(*)$, but with ϕ and ψ interchanged: $\phi \leftrightarrow \psi$

$$\int_V \left[\vec{\nabla}\phi\cdot\vec{\nabla}\psi + \psi\,\vec{\nabla}^2\phi\right] d^3x = \oint_S \psi\,\frac{\partial\phi}{\partial n}\, da \qquad (**)$$

[i.e., for these same ϕ and ψ, if we defined $\vec{Z} = \psi\vec{\nabla}\phi$ and applied the div. thm. to $\vec{Z}$]

Now, subtract (**) from (*):

$$\boxed{\int_V \left[\phi \vec{\nabla}^2 \psi - \psi \vec{\nabla}^2 \phi \right] d^3x = \oint_S \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] da}$$

Green's Theorem / Green's Second Identity

What can we do with this as is relevant for electrostatics? Convert Poisson to $\int$ eq'n

Let: $\phi = \Phi$ which obeys $\vec{\nabla}^2 \Phi = -\rho/\varepsilon_0$)

$\psi = \frac{1}{R} \equiv \frac{1}{|\vec{x} - \vec{x}'|}$ $\vec{x}$: "observation point", $\vec{x}'$: "integration variable"

We already know that:

$$\vec{\nabla}^2 \frac{1}{|\vec{x} - \vec{x}'|} = \vec{\nabla}^2 \psi = -4\pi \delta(\vec{x} - \vec{x}')$$

Substituting this into Green's Theorem gives us:

$$\int_V \left[-\Phi(\vec{x}') \cdot 4\pi \delta(\vec{x} - \vec{x}') + \frac{1}{R} \cdot \frac{\rho(\vec{x}')}{\varepsilon_0} \right] d^3x' =$$

$$\oint_S \left[\Phi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) - \frac{1}{R} \frac{\partial \Phi}{\partial n'} \right] da'$$

So it follows from the properties of the δ-function that <u>if the point $\vec{x}$ is within the volume V</u>:

$$-4\pi \Phi(\vec{x}) + \int_V \frac{1}{R} \frac{\rho(\vec{x}')}{\varepsilon_0} d^3x' = \oint_S \left[\Phi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) - \frac{1}{R} \frac{\partial \Phi}{\partial n'} \right] da'$$

$$\Rightarrow \boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\vec{x}')}{R} d^3x' + \frac{1}{4\pi} \oint_S \left[\frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) \right] da'}$$

If the point $\vec{x}$ is <u>NOT</u> within V, then integration of the δ-function gives 0,

$$0 = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\vec{x}')\, d^3x'}{R} + \frac{1}{4\pi} \oint_S \left[\frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) \right] da'$$