Let's interpret this: 

① if surface $S \to \infty$ and $\frac{1}{R} \cdot \frac{\partial \Phi}{\partial n'}$ falls faster than $\frac{1}{R^2}$ (i.e., faster than the surface $\propto R^2$) $\Rightarrow$ surface integral $\to 0$ [assuming $\Phi(\infty) \to 0$]

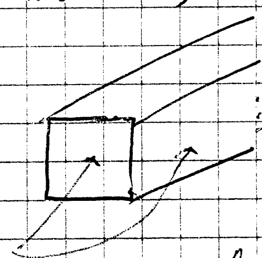
recover familiar result: $\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')\, d^3x'}{|\vec{x} - \vec{x}'|}$

② if $\rho = 0$ (no charge, period) inside the volume V, $\Phi$ can be specified anywhere inside V in terms of the potential $\Phi$ and $\frac{\partial \Phi}{\partial n}$ (normal derivative) only on the surface (S) of V.

This specification of both $\Phi$ and $\frac{\partial \Phi}{\partial n}$ on the surface is termed "Cauchy boundary conditions" and, is an "integral statement", not a boundary value problem (i.e., doesn't display the power of the method!!)

Dirichlet and Neumann Boundary Conditions
(b.c.'s appropriate for solution of Poisson equations)
Example of a "boundary value problem":

(closed box)



specify potential on surface (Volts)
what is solution for $\Phi$ inside the box?

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① Dirichlet boundary condition: specification of the potential on a closed surface defines a unique potential problem

② Neumann boundary conditions: specification of the normal derivative of the potential $\frac{\partial \Phi}{\partial n}$ (i.e., $\vec{E}$ field)

Question: Are these appropriate for the Poisson or Laplace equation to ensure that a unique/well-behaved solution exists inside the bounded region?

Let's prove the uniqueness of the solution to the Poisson equation, (next time: solution)

$$\vec{\nabla}^2 \Phi = -\rho/\varepsilon_0$$

inside a volume $V$ subject to either Dirichlet or Neumann boundary conditions on the closed bounding surface $S$.

Suppose we claim that there exist 2 independent solutions, $\Phi_1$ and $\Phi_2$, both of which satisfy the boundary conditions and Poisson equation. Then define:

$$\mathcal{U} = \Phi_2 - \Phi_1 ,$$

where: $\vec{\nabla}^2 \Phi_1 = \vec{\nabla}^2 \Phi_2 = -\rho/\varepsilon_0$ inside $V$

$$\Rightarrow \quad \vec{\nabla}^2 \mathcal{U} = 0$$

s.t. s.t.

$\Phi_1(S) = \Phi_2(S)$ $\quad \mathcal{U} = 0$ or $\dfrac{\partial \mathcal{U}}{\partial n} = 0$

Dirichlet $\qquad$ Neumann (as both satisfy the same $\dfrac{\partial \Phi_1}{\partial n} = \dfrac{\partial \Phi_2}{\partial n}$ b.c.)

Recall Green's First Identity:

$$\int_V \left[ \phi \vec{\nabla}^2 \psi + \vec{\nabla}\phi \cdot \vec{\nabla}\psi \right] d^3x' = \oint_S \phi \frac{\partial \psi}{\partial n'} \, da'$$

Set: $\phi = \psi = \mathcal{U}$

$$\Rightarrow \int_V \left[ \underbrace{\mathcal{U} \vec{\nabla}^2 \mathcal{U}}_{= 0 \text{ by construction!}} + \vec{\nabla}\mathcal{U} \cdot \vec{\nabla}\mathcal{U} \right] d^3x' = \boxed{\oint_S \mathcal{U} \frac{\partial \mathcal{U}}{\partial n'} \, da'}$$

Dirichlet: $\mathcal{U} = 0$ on $S$ $\Rightarrow \displaystyle\int_V \left[ \vec{\nabla}\mathcal{U} \cdot \vec{\nabla}\mathcal{U} \right] d^3x' = \int_V |\vec{\nabla}\mathcal{U}|^2 d^3x' = 0$

Neumann: $\dfrac{\partial \mathcal{U}}{\partial n} = 0$ on $S$ $\Rightarrow \displaystyle\int_V |\vec{\nabla}\mathcal{U}|^2 d^3x' = 0$

[as $|\vec{\nabla}\mathcal{U}|^2 \geq 0$]

$$\Rightarrow \underbrace{\vec{\nabla}\mathcal{U} = 0}_{\text{over } V} \Rightarrow \mathcal{U} = \text{constant inside } V$$

- For Dirichlet: $\mathcal{U} = 0$ on $S$ $\Rightarrow \Phi_1 = \Phi_2$ ✓

$C_i$: integration constants

- For Neumann: $\dfrac{\partial \mathcal{U}}{\partial n} = 0$ on $S \Rightarrow \dfrac{\partial \Phi_1}{\partial n} = \dfrac{\partial \Phi_2}{\partial n} \Rightarrow \Phi_1 + C_1 = \Phi_2 + C_2$

$\Rightarrow$ identical up to unimportant constant

## Green Function

Now: desire solution to Poisson or Laplace equation in a finite volume V with either Dirichlet or Neumann boundary conditions on the surface S via Green's theorem and Green functions

Last Time: showed solution satisfying D or N b.c.'s will be unique! to Poisson/Laplace

Note, previously obtained:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{|\vec{x}-\vec{x}'|}\frac{\partial\Phi}{\partial n'} - \underset{(\vec{x}')}{\Phi}\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]da'$$

[This is NOT a "solution" for $\Phi(\vec{x})$. Cannot specify both $\Phi$ and $\frac{\partial\Phi}{\partial n}$ on S arbitrarily $\Rightarrow$ as demonstrated, unique solutions for Dirichlet and Neumann conditions separately; and in general, will not be consistent w/ specification of both $\Phi$ and $\partial_n\Phi$]

Recall in obtaining this, we set $\psi = \frac{1}{R} = \frac{1}{|\vec{x}-\vec{x}'|}$ in Green's Theorem. This satisfies:

$$\underbrace{\vec{\nabla}'^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)} = -4\pi\,\delta(\vec{x}-\vec{x}')$$

[derivative w.r.t. the ' variables, which are being integrated over]

analogy: $\vec{\nabla}^2\Phi = -\rho/\varepsilon_0 \Rightarrow \propto$ potential of a point charge source

In general, $\frac{1}{|\vec{x}-\vec{x}'|}$ is a member of a class of functions which satisfy, in general:

$$\vec{\nabla}'^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$$

$$\text{where } G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + \underbrace{F(\vec{x},\vec{x}')}$$

satisfies $\vec{\nabla}'^2 F(\vec{x},\vec{x}') = 0$ inside volume V

General Idea:

Use Green's Theorem with $\psi = G(\vec{x},\vec{x}')$ and choose $F(\vec{x},\vec{x}')$ to eliminate one or the other of the two surface integrals in Green's Theorem

thus obtaining a result that involves only Dirichlet or Neumann boundary conditions.