as $\vec{\nabla}'^2 F = 0$ by construction

Doing so with: $\phi = \Phi$, $\psi = G(\vec{x},\vec{x}')$, and using $\vec{\nabla}'^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$ (25)

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x' + \frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{x}')\frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} \right] da'$$

For Dirichlet Boundary Conditions, demand:

$$G_D(\vec{x},\vec{x}') = 0 \quad \text{for } \vec{x}' \text{ on } S$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\,G_D(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\,\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}\,da'$$

For Neumann Boundary Conditions: (careful!!)

Suppose we specified:

$$\frac{\partial G_N}{\partial n'}(\vec{x},\vec{x}') = 0 \quad \text{for } \vec{x}' \text{ on } S$$

(would eliminate second surface integral)

But this is inconsistent with:

$$\vec{\nabla}'^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$$

By Gauss's / Divergence Theorem:

$$\int_V \vec{\nabla}'\cdot(\vec{\nabla}' G)\,d^3x' = \oint_S (\vec{\nabla}' G)\cdot d\vec{a}', \quad d\vec{a}' = da'\,\hat{n}$$

$$\Rightarrow \vec{\nabla}' G \cdot d\vec{a}' = \frac{\partial G}{\partial n'}\,da'$$

$$-\int 4\pi\,\delta(\vec{x}-\vec{x}')\,d^3x' = \oint_S \frac{\partial G}{\partial n'}\,da'$$

$$-4\pi = \oint_S \frac{\partial G}{\partial n'}\,da',$$

so $\frac{\partial G}{\partial n'} = 0$ on S not allowed

Instead, CAN specify:

$$\frac{\partial G_N(\vec{x},\vec{x}')}{\partial n'} = \frac{-4\pi}{S}$$

S: total area of bounding surface

this satisfies: $-4\pi = \oint_S -\frac{4\pi}{S}\,da' = \frac{-4\pi}{S}\oint_S da' = -4\pi$ ✓

Then we have for $\Phi(\vec{x})$:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\, G_N(\vec{x},\vec{x}')\, d^3x' + \frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'}\, da'$$

$$- \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\left(-\frac{4\pi}{S}\right) da'$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\, G_N(\vec{x},\vec{x}')\, d^3x' + \frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'}\, da'$$

$$+ \langle\Phi\rangle_S$$

average value of Φ on the surface S

[Recall in 1-D: $\langle f(x)\rangle = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)\, dx$

or 2-D: $\langle f(x,y)\rangle = \frac{1}{(\text{area})}\iint_{\text{area}} dx\, dy\, f(x,y)$

What is the physical meaning of $F(\vec{x},\vec{x}')$?

Well, $F(\vec{x},\vec{x}')$ satisfies:

$$\vec{\nabla}'^2 F(\vec{x},\vec{x}') = 0$$

Recall, $\vec{\nabla}^2\Phi = -\rho/\epsilon_0 \Rightarrow$ $F(\vec{x},\vec{x}')$ is a solution of the Laplace equation (i.e., $\rho = 0$) inside V and so represents the potential (if any) due to a system of charges <u>external</u> to the volume V

So, $F(\vec{x},\vec{x}')$ can be thought of as the potential due to an external system of charges chosen to satisfy the boundary conditions either on Φ (Dirichlet) or on $\frac{\partial\Phi}{\partial n'}$ (Neumann), when combined with the "true" charges at the $\vec{x}'$ points.

$\Rightarrow$ "well, this is equivalent to the "method of images".

Electrostatic Potential Energy

We discussed a short while ago that the work done on a charge q_i to bring it from ∞ to $\vec{x}_i$ is:

$$W_i = q_i \Phi(\vec{x}_i) \Rightarrow \text{thus, its potential energy}$$

Now, if Φ is due to a system of $N-1$ charges q_j at positions $\vec{x}_j$, it follows that:

$$\Phi(\vec{x}_i) = \frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{N-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|}$$

so the potential energy of charge q_i is:

$$W_i = \frac{q_i}{4\pi\varepsilon_0} \sum_{j=1}^{N-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|}$$

Now, if we consider the total potential energy of all of the N charges due to all of the Coulomb forces acting between them:

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \sum_{j>i} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

(enforces no "self-energy" terms and double counting)

e.g., if $\exists$ 3 charges, (q_1, q_2, q_3) at $(\vec{x}_1, \vec{x}_2, \vec{x}_3)$:

$$W = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|} + \frac{q_1 q_3}{|\vec{x}_1 - \vec{x}_3|} + \frac{q_2 q_3}{|\vec{x}_2 - \vec{x}_3|} \right] \checkmark$$

More symmetric form: let i, j sums run over all N:

$$W = \frac{1}{8\pi\varepsilon_0} \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N}}_{i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

(factor of $\frac{1}{2}$ to account for double counting)

Same e.g.)

$$W = \frac{1}{8\pi\varepsilon_0} \left[\frac{q_1 q_2}{|\ldots|} + \frac{q_1 q_3}{|\ldots|} + \frac{q_2 q_1}{|\ |} + \frac{q_2 q_3}{|\ldots|} + \frac{q_3 q_1}{|\ .\ |} + \frac{q_3 q_2}{|\ldots|} \right] \checkmark$$