Electrostatic Potential Energy

We discussed a short while ago that the work done on a charge q_i to bring it from ∞ to $\vec{x}_i$ is:

$$W_i = q_i \Phi(\vec{x}_i) \Rightarrow \text{thus, its potential energy}$$

Now, if Φ is due to a system of $N-1$ charges q_j at positions $\vec{x}_j$, it follows that:

$$\Phi(\vec{x}_i) = \frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{N-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|}$$

so the potential energy of charge q_i is:

$$W_i = \frac{q_i}{4\pi\varepsilon_0} \sum_{j=1}^{N-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|}$$

Now, if we consider the total potential energy of all of the N charges due to all of the Coulomb forces acting between them:

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \sum_{j>i} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

(enforces no "self-energy" terms and double counting)

e.g., if $\exists$ 3 charges, (q_1, q_2, q_3) at $(\vec{x}_1, \vec{x}_2, \vec{x}_3)$:

$$W = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|} + \frac{q_1 q_3}{|\vec{x}_1 - \vec{x}_3|} + \frac{q_2 q_3}{|\vec{x}_2 - \vec{x}_3|}\right] \checkmark$$

More symmetric form: let i, j sums run over all N:

$$W = \frac{1}{8\pi\varepsilon_0} \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N}}_{i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

(factor of $\frac{1}{2}$ to account for double counting)

Same e.g.:

$$W = \frac{1}{8\pi\varepsilon_0}\left[\frac{q_1 q_2}{|\ldots|} + \frac{q_1 q_3}{|\ldots|} + \frac{q_2 q_1}{|\ |} + \frac{q_2 q_3}{|\ldots|} + \frac{q_3 q_1}{|\ldots|} + \frac{q_3 q_2}{|\ldots|}\right] \checkmark$$

If charge distribution is continuous, instead of discrete point charges:

$$W = \frac{1}{8\pi\varepsilon_0} \int\int_{\text{all space}} \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x\, d^3x'$$

But, recall: $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$

$$\Rightarrow W = \frac{1}{2}\int_{\text{all space}} \rho(\vec{x})\,\Phi(\vec{x})\, d^3x$$

} electrostatic potential in terms of the positions $\vec{x}$ of the charges in some potential
$\Rightarrow$ Coulomb force interactions

Recall, alternatively, undergrad E+M, learned about the energy stored in the $\vec{E}$ field.

$$\vec{\nabla}^2\Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\varepsilon_0}$$

$$\Rightarrow W = \frac{-\varepsilon_0}{2}\int \left(\vec{\nabla}^2\Phi(\vec{x})\right)\cdot\Phi(\vec{x})\, d^3x$$

$$= -\frac{\varepsilon_0}{2}\int \Phi(\vec{x})\cdot\vec{\nabla}^2\Phi(\vec{x})\cdot d^3x = -\frac{\varepsilon_0}{2}\int\left[\Phi(\vec{\nabla}\cdot\vec{\nabla}\Phi\right) d^3x$$

Integrate by parts: (in 3 dimension) $\left[\begin{array}{l}\vec{\nabla}\cdot(\psi\vec{a}) = \vec{a}\cdot\vec{\nabla}\psi + \psi\vec{\nabla}\cdot\vec{a} \\ \Rightarrow \int_V [\vec{a}\cdot\vec{\nabla}\psi + \psi\vec{\nabla}\cdot\vec{a}]\,d^3x = \oint_S \psi\vec{a}\cdot\hat{n}\,da \\ \text{by Div. Thm}\end{array}\right]$

$$\int_V \vec{\nabla}u\cdot\vec{V}\, d^3x = \oint_S (u\vec{V})\cdot\hat{n}\, da - \int_V u(\vec{\nabla}\cdot\vec{V})\, d^3x$$

$$\left[\begin{array}{l} u = \frac{\varepsilon_0}{2}\Phi \\ \vec{V} = \vec{\nabla}\Phi \end{array}\right]$$

$$\Rightarrow W = \frac{\varepsilon_0}{2}\int_{\text{all space}} (\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi)\, d^3x \quad - \underbrace{\frac{\varepsilon_0}{2}\oint_{S\text{ at }\infty} \Phi(\vec{\nabla}\Phi\cdot\hat{n})\, da}$$

} integrals are over all space so at ∞, $\Phi \to 0$

$$\vec{E} = -\vec{\nabla}\Phi$$

$$W = \frac{\varepsilon_0}{2}\int (-\vec{E})\cdot(-\vec{E})\, d^3x$$

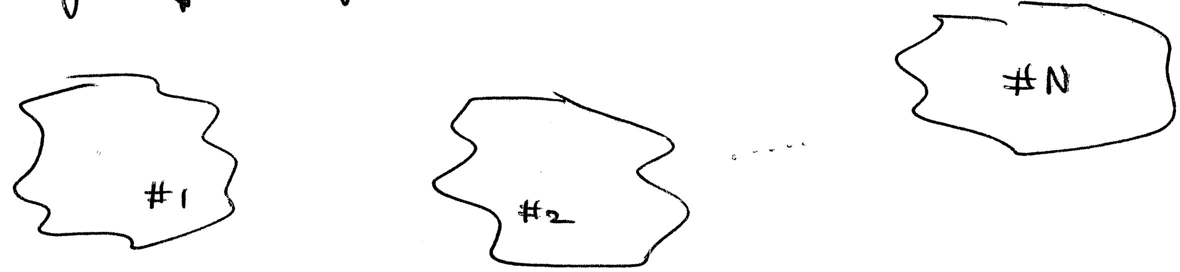
$$= \frac{\varepsilon_0}{2}\int |\vec{E}|^2 d^3x$$

Define energy density w:

$$w \equiv \frac{\varepsilon_0}{2}|\vec{E}|^2 \Rightarrow W = \int w\, d^3x$$

Capacitance:

Consider system of N conductors:



[By uniqueness theorem, if potential specified on each conductor, ∃ only one unique distribution of charges which yields these potentials!]

Place positive unit charge on #1, but leave others uncharged. This produces potentials:

$P_{11}, P_{21}, \ldots, P_{N1}$ on the N conductors.

Similarly, placing positive unit charge on n^{th} conductor, leaving others uncharged, generates potentials:

$P_{1n}, P_{2n}, \ldots, P_{Nn}$ on the N conductors [P_{in} depend on geometry of problem]

So, in general, placing Q_n on #n, maintaining others uncharged, gives rise to:

$P_{1n} Q_n, P_{2n} Q_n, \ldots, P_{Nn} Q_n$ on the N conductors

Superpose these distributions:

Effect of $Q_1, Q_2, \ldots, Q_N$ on N conductors

$$V_1 = P_{11} Q_1 + P_{12} Q_2 + \ldots + P_{1N} Q_N$$
$$V_2 = P_{21} Q_1 + P_{22} Q_2 + \ldots + P_{2N} Q_N$$
$$\vdots \qquad\qquad \vdots$$

$$\Rightarrow \boxed{V_i = \sum_{j=1}^{N} P_{ij} Q_j}$$

Capacitance:

⇒ For our system of conductors:

$$V_i = \sum_{j=1}^{N} p_{ij}\, Q_j$$

p_{ij}: depend on the geometry of the conductors (result from doing the integral)

Inverting this: (above matrix equation)

$$\begin{pmatrix} V_1 \\ V_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \vdots \end{pmatrix}$$

$$Q_i = \sum_{j=1}^{N} C_{ij} V_j$$

C_{ii} : "capacitances"

$C_{ij},\ i \neq j$: "coefficients of induction"

So what we see is: $\underbrace{Q_i}_{\text{total charge}} = C_{ii} \underbrace{V_i}_{\text{potential}}$ (all other conductors at zero potential $j \neq i$)

Consider case where all conductors at zero potential except for i^{th} conductor

Potential energy of a system of conductors:

$$W = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x \quad \text{(continuous)}$$

$$\left[\text{if: } \rho(\vec{x}) \cdot \Phi(\vec{x}) = \sum_{i=1}^{N} Q_i\, V_i\, \delta(\vec{x}_i - \vec{x}) \right] \quad \text{"point conductors"}$$

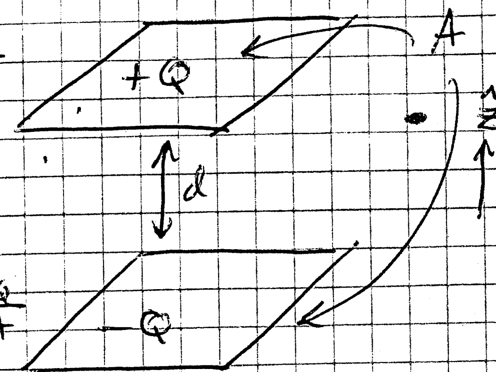
$$\Rightarrow W = \frac{1}{2} \sum_{i=1}^{N} Q_i (V_i) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (V_i) \cdot C_{ij} V_j$$

Example i [Jackson 1.6, 1.8, 1.9]

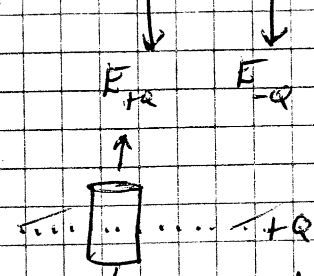
(1.6) Use Gauss's law to calculate the capacitance of two large, flat conducting sheets of area A, separated by a small distance d. Assume equal/opposite charges on the conductors.
($d << A$, so like infinite sheets) ignore "end effects", consider later w/ conformal mapping

$\sigma_+ = \frac{Q}{A}$

$\sigma_- = \frac{-Q}{A}$



In region between the plates,



$[\vec{E} = 0 \text{ elsewhere}]$

By Gauss's Law: $2|\vec{E}_+|\,da = \frac{1}{\varepsilon_0}\frac{Q}{A}\cdot da$

$$\Rightarrow |\vec{E}_+| = \frac{Q}{2\varepsilon_0 A}$$

Similarly, $|\vec{E}_-| = \frac{Q}{2\varepsilon_0 A}$

$$\Rightarrow |\vec{E}| = \frac{Q}{\varepsilon_0 A}$$

$$\vec{E} = -\frac{Q}{\varepsilon_0 A}\hat{z}, \quad d\vec{l} = dz\,\hat{z}$$

$$|V| = \left|-\int \vec{E}\cdot d\vec{l}\right| = \frac{Q}{\varepsilon_0 A}\cdot d \Rightarrow C = \frac{Q}{V} = \frac{Q}{\frac{Qd}{\varepsilon_0 A}} = \boxed{\frac{\varepsilon_0 A}{d}} \checkmark$$

(1.8) Calculate the total electrostatic energy of this capacitor geometry; and the electrostatic energy density of the field.

$$W = \frac{\varepsilon_0}{2}\int |\vec{E}|^2 d^3x, \quad [\vec{E} = 0 \text{ outside region in between plates!}]$$

$$= \frac{\varepsilon_0}{2}\int \frac{Q^2}{\varepsilon_0^2 A^2} d^3x = \frac{Q^2}{2\varepsilon_0 A^2}\cdot \underbrace{\int d^3x}_{\text{(volume between plates)}} = \frac{Q^2}{2\varepsilon_0 A^2}\cdot A d = \frac{Q^2 d}{2\varepsilon_0 A}$$

Note: $W = \frac{1}{2}\frac{Q^2 d}{\varepsilon_0 A} = \frac{1}{2}\cdot \underbrace{\left(\frac{\varepsilon_0 A}{d}\right)}_{=C} \underbrace{\left(\frac{Q^2 d^2}{\varepsilon_0^2 A^2}\right)}_{V^2}$ ✓

$w = \frac{\varepsilon_0}{2}|\vec{E}|^2 = \frac{\varepsilon_0}{2}\cdot\frac{Q^2}{\varepsilon_0^2 A^2} = \frac{Q^2}{2\varepsilon_0 A^2}$, check: $w \cdot Ad = \frac{Q^2 d}{2\varepsilon_0 A}$ ✓

(1.9) Attractive force between conductors?

$$|\vec{F}| = |Q\vec{E}| = Q\cdot\frac{Q}{2\varepsilon_0 A} = \frac{Q^2}{2\varepsilon_0 A} \checkmark$$

from other charge only!! (does not exert force on itself!!)