Method of Images

- closely related to the use of Green functions for the solution of electrostatic boundary value problems

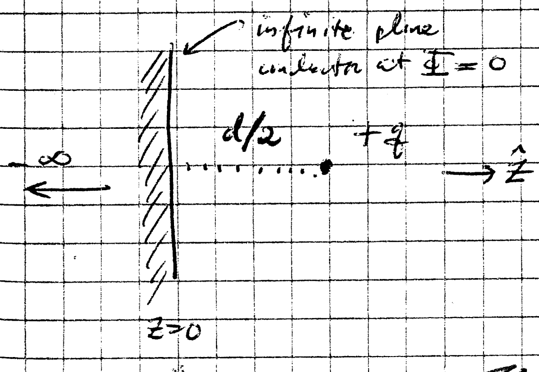
- one or more point charges in the presence of boundary surfaces (e.g., conductors grounded or at fixed potentials)

$\Rightarrow$ ideally: small number of suitably-placed charges, "image charges", external to the region of interest, simultaneously simulate the boundary conditions while still satisfying Laplace Equation inside region of interest [they do not add a $\rho(\vec{x})$ inside]

"method of images": replacement of actual problem with boundaries and their potentials as required by an enlarged region with image charges but no boundaries!!

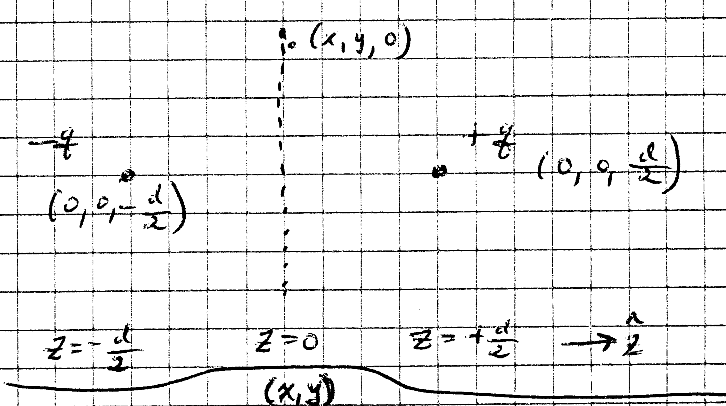
Classic Problem: What is the distribution of charge on ..... ? [best illustrated via examples]

Original Boundary-Value Problem:



$\bigstar$ potential problem for $z>0$ half space

Image Problem:

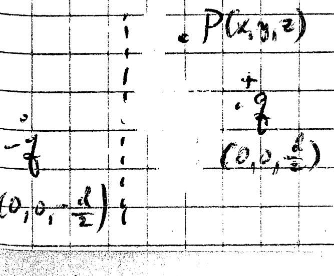


Clearly, for any (x,y) point on the $z=0$ plane,

$$\Phi = \frac{+q}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2+(\frac{d}{2})^2}} + \frac{-q}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2+(\frac{d}{2})^2}}$$

$= 0$ ✓ (so b.c. is satisfied) for all (x,y) on $z=0$ plane

Now, for our image problem:



At any point P above the plane $(z>0)$

$$\vec{E}(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{x\hat{x} + y\hat{y} + (z-\frac{d}{2})\hat{z}}{[x^2+y^2+(z-\frac{d}{2})^2]^{3/2}}\right] + - \frac{q}{4\pi\varepsilon_0}\left[\frac{x\hat{x} + y\hat{y} + (z+\frac{d}{2})\hat{z}}{[x^2+y^2+(z+\frac{d}{2})^2]^{3/2}}\right]$$

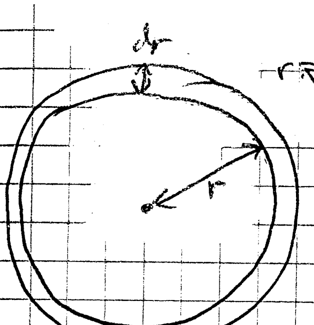
On the plane ($z=0$):

$$\vec{E}_0 = \frac{q}{4\pi\varepsilon_0}\left[\frac{x\hat{x}+y\hat{y}-\frac{d}{2}\hat{z}}{(x^2+y^2+d^2/4)^{3/2}}\right] - \frac{q}{4\pi\varepsilon_0}\left[\frac{x\hat{x}+y\hat{y}+\frac{d}{2}\hat{z}}{(x^2+y^2+d^2/4)^{3/2}}\right]$$

$$= -\frac{q}{4\pi\varepsilon_0}\frac{d}{(x^2+y^2+d^2/4)^{3/2}}\hat{z}$$ } $\vec{E}$ normal to conductor surface everywhere for all (x,y)! ✓

To find surface charge, use Gauss's Law for "original problem" (now that method of images yielded $\vec{E}$)

Looking down onto the conductor:



$\odot\hat{z}$

$\otimes\vec{E}_0$

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$$

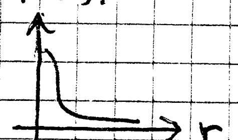
We see $\vec{E}_0 = \frac{-q}{4\pi\varepsilon_0}\frac{d}{(r^2+\frac{d^2}{4})^{3/2}}\hat{z}$

At some r: (annulus) $\leftrightarrow$ "pillbox"

$$\left(\frac{-q}{4\pi\varepsilon_0}\frac{d}{(r^2+\frac{d^2}{4})^{3/2}}\hat{z}\right)\cdot(da\,\hat{z}) = \frac{\sigma(r)\cdot da}{\varepsilon_0}$$

$[da = 2\pi r\,dr]$

$$\Rightarrow \sigma(r) = -\frac{q}{4\pi}\frac{d}{(r^2+d^2/4)^{3/2}}$$

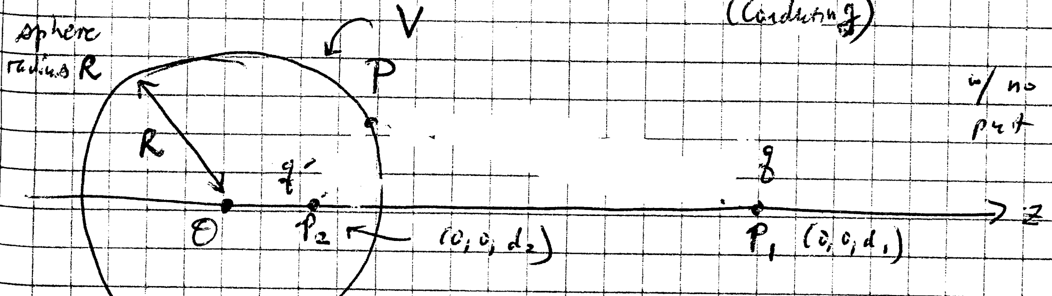


What is the total induced surface charge on the conductor?

$$Q_{cond} = \int_0^\infty \sigma(r)(2\pi r\,dr) = -\frac{qd}{4\pi}2\pi\int_0^\infty\frac{r}{(r^2+\frac{d^2}{4})^{3/2}}dr$$

$$= -\frac{qd}{2}\left[-\left(r^2+\frac{d^2}{4}\right)^{-1/2}\right]\Bigg|_{r=0}^{r=\infty} = -\frac{qd}{2}\left[0+\frac{1}{\frac{d}{2}}\right] = -q \checkmark$$

Another Classic Problem: Point Charge Near Grounded Sphere (at Fixed Potential $V=0$) (Conducting)



w/ no loss of generality, put $+q$ along z-axis

By symmetry: image charge q' must lie along z. If q outside sphere, q' inside sphere.

Seek potential Φ s.t. Φ(sphere surface) $= 0$ //

Potential due to q and q' at P is: V

point on sphere centered on origin: in spherical coordinates, (r, θ, ϕ)

$$P: \begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases} \qquad P_1: \overset{x\ y\ z}{(0, 0, d_1)} \qquad P_2: \overset{x\ y\ z}{(0, 0, d_2)}$$

$$\Phi(P) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left[R^2\sin^2\theta\cos^2\phi + R^2\sin^2\theta\sin^2\phi + (R\cos\theta - d_1)^2\right]^{1/2}} +$$

$$\frac{q'}{4\pi\epsilon_0}\frac{1}{\left[R^2\sin^2\theta\cos^2\phi + R^2\sin^2\theta\sin^2\phi + (R\cos\theta - d_2)^2\right]^{1/2}} = 0$$

$$\Rightarrow 0 = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{R^2 - 2d_1R\cos\theta + d_1^2}} + \frac{q'}{4\pi\epsilon_0}\frac{1}{\sqrt{R^2 - 2d_2R\cos\theta + d_2^2}}$$

$$\Rightarrow 0 = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{1 - 2\left(\frac{d_1}{R}\right)\cos\theta + \left(\frac{d_1}{R}\right)^2}} + \frac{q'}{\sqrt{1 - 2\left(\frac{d_2}{R}\right)\cos\theta + \left(\frac{d_2}{R}\right)^2}}\right]$$

$$\Rightarrow 0 = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{1 - 2\left(\frac{d_1}{R}\right)\cos\theta + \left(\frac{d_1}{R}\right)^2}} + \frac{q'}{\left(\frac{d_2}{R}\right)}\frac{1}{\sqrt{1 - 2\left(\frac{R}{d_2}\right)\cos\theta + \left(\frac{R}{d_2}\right)^2}}\right]$$

We see this can be satisfied if:

$$\frac{d_1}{R} = \frac{R}{d_2} \quad \text{and} \quad q + \frac{q'}{\left(\frac{d_2}{R}\right)} = 0$$

$$d_2 = \frac{R^2}{d_1} \qquad\qquad q' = -q\left(\frac{d_2}{R}\right) = -q\left(\frac{R}{d_1}\right)$$

- As q is brought towards the sphere, q' increases in magnitude
  $d_2 \rightarrow$ moves towards edge of sphere
  (if $d_1 >> R$, $d_2 << R$)
- If q is just outside the sphere: ($d_1 \cong R + \epsilon$)
  $q' \approx -q$, $d_2 \cong R$ ✓

So now we have a solution for Φ due to q and q':

For arbitrary (r, θ, ϕ):

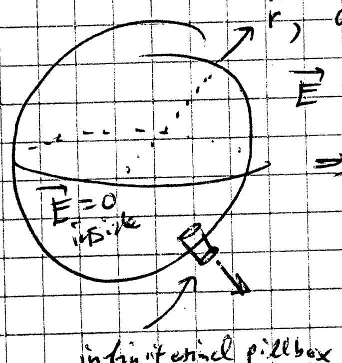
$$\Phi(r,\theta,\phi) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2 - 2d_1 r\cos\theta + d_1^2}} + \frac{q'}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2 - 2d_2 r\cos\theta + d_2^2}}$$

with: $d_2 = R^2/d_1$, $q' = -q\left(\frac{R}{d_1}\right)$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2 - 2d_1 r\cos\theta + d_1^2}} - \frac{q}{4\pi\varepsilon_0}\left(\frac{R}{d_1}\right)\cdot\frac{1}{\sqrt{r^2 - \frac{2rR^2}{d_1}\cos\theta + \frac{R^4}{d_1^2}}}$$

Now, find charge density σ on sphere:

By Gauss's Law:



$\hat{r}$, $d\vec{a} = da\,\hat{r}$

$\vec{E} = E_r\hat{r}$ right on the surface of the conductor!

$$\Rightarrow \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$E_r\cdot da\Big|_{\substack{S\\(r=R)}} = \frac{1}{\varepsilon_0}\sigma\cdot da$$

in spherical, $\vec{\nabla}\psi = \partial_r\psi\,\hat{r} + \frac{1}{r}\partial_\theta\psi\,\hat{\theta} + \frac{1}{r\sin\theta}\partial_\phi\psi\,\hat{\phi}$

$$\Rightarrow E_r = -\vec{\nabla}\Phi\cdot\hat{r} = -\partial_r\Phi$$

$$\frac{\partial\Phi}{\partial r} = \frac{q}{4\pi\varepsilon_0}\left[r^2 - 2d_1 r\cos\theta + d_1^2\right]^{-3/2}\left(-\frac{1}{2}\right)\cdot(2r - 2d_1\cos\theta)$$

$$- \frac{q}{4\pi\varepsilon_0}\left(\frac{R}{d_1}\right)\left(-\frac{1}{2}\right)\left[r^2 - \frac{2rR^2}{d_1}\cos\theta + \frac{R^4}{d_1^2}\right]^{-3/2}\cdot\left(2r - \frac{2R^2}{d_1}\cos\theta\right)$$

At $r = R$, (on S)

$$\frac{\partial\Phi}{\partial r} = -\frac{q}{8\pi\varepsilon_0}\frac{2R - 2d_1\cos\theta}{\left[R^2 - 2d_1 R\cos\theta + d_1^2\right]^{3/2}} + \frac{q}{8\pi\varepsilon_0}\left(\frac{R}{d_1}\right)\frac{2R - \frac{2R^2}{d_1}\cos\theta}{\left[R^2 - \frac{2R^3}{d_1}\cos\theta + \frac{R^4}{d_1^2}\right]^{3/2}}$$

$$= \frac{-q}{8\pi\varepsilon_0}\frac{2R - 2d_1\cos\theta}{\left[R^2 - 2d_1 R\cos\theta + d_1^2\right]^{3/2}} + \frac{q\left(\frac{R}{d_1}\right)}{8\pi\varepsilon_0}\frac{2R - 2\frac{R^2}{d_1}\cos\theta}{\left[\frac{R^2}{d_1^2}\left(R^2 + d_1^2 - 2\frac{R d_1}{}\cos\theta\right)\right]^{3/2}}$$

$$= \frac{-q}{8\pi\varepsilon_0}\frac{2R - 2d_1\cos\theta}{\left[R^2 - 2d_1 R\cos\theta + d_1^2\right]^{3/2}} + \frac{q}{8\pi\varepsilon_0}\cdot\frac{2\frac{d_1^2}{R} - 2d_1\cos\theta}{\left[R^2 - 2Rd_1\cos\theta + d_1^2\right]^{3/2}}$$

$$= \frac{q}{8\pi\varepsilon_0}\frac{1}{\left[R^2 - 2d_1 R\cos\theta + d_1^2\right]^{3/2}}\left[\frac{2d_1^2}{R} - 2R\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{d_1^3}\frac{1}{\left[1 + \frac{R^2}{d_1^2} - 2\frac{R}{d_1}\cos\theta\right]^{3/2}}\cdot\left(\frac{d_1^2}{R} - R\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\left[1 + \frac{R^2}{d_1^2} - 2\frac{R}{d_1}\cos\theta\right]^{3/2}}\frac{1}{d_1^2}\cdot\left(\frac{d_1}{R} - \frac{R}{d_1}\right)$$

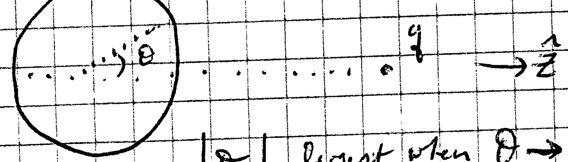
$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\left[1 + \frac{R^2}{d_1^2} - 2\frac{R}{d_1}\cos\theta\right]^{3/2}}\frac{1}{d_1^2}\cdot\frac{R}{d_1}\left(-1 + \frac{d_1^2}{R^2}\right)$$

$$\Rightarrow \sigma\Big|_S = \varepsilon_0 E_r\Big|_S = \varepsilon_0\left(-\partial_r\Phi\right)\Big|_S$$

$$= \frac{q}{4\pi}\frac{1}{\left[1 + \left(\frac{R^2}{d_1^2}\right) - 2\left(\frac{R}{d_1}\right)\cos\theta\right]^{3/2}}\cdot\frac{1}{R^2}\cdot\frac{1}{d_1^2}\cdot\frac{R^3}{d_1}\left(1 - \frac{d_1^2}{R^2}\right)$$

$$= \underbrace{-\frac{q}{4\pi R^2}\frac{1}{\left[1 + \left(\frac{R^2}{d_1^2}\right) - 2\left(\frac{R}{d_1}\right)\cos\theta\right]^{3/2}}\cdot\left(\frac{R}{d_1}\right)\cdot\left(1 - \frac{R^2}{d_1^2}\right)}_{\text{can prove total induced charge on } S \text{ is just } q' \text{ via direct integration}} \checkmark$$

See 34b



$|\sigma|$ largest when $\theta \to 0$, smallest when $\theta \to \pi$ $\Big\}$ "induced charge" greatest on cap near the charge q,

What is the force on the charge q?

Easiest way via Coulomb's law for q and q':
(attractive force!) + and −

$$|\vec{F}| = \frac{1}{4\pi\varepsilon_0}\cdot q\left(q\frac{R}{d_1}\right)\cdot\frac{1}{(d_1 - d_2)^2}$$

$$(d_1 - d_2)^2 = \left(d_1 - \frac{R^2}{d_1}\right)^2 = \left[d_1\left(1 - \frac{R^2}{d_1^2}\right)\right]^2$$

$$= \frac{q^2}{4\pi\varepsilon_0}\frac{R}{d_1}\cdot\frac{1}{d_1^2}\frac{1}{\left[1 - R^2/d_1^2\right]^2}$$

$$= \frac{q^2}{4\pi\varepsilon_0}\frac{1}{R^2}\cdot\frac{R^3}{d_1^3}\frac{1}{\left[1 - R^2/d_1^2\right]^2}$$

For $d_1 \to \infty$, $|\vec{F}| \propto \frac{1}{d_1^3}$

Integrate σ_s over surface of the sphere:

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \cdot R^2 \cdot \sigma_s$$

$$= -\cancel{2\pi} \cancel{R^2} \int_0^{\pi} \sin\theta \, d\theta \, \frac{q}{\cancel{4\pi R^2}} \frac{1}{\left[1 + \left(\frac{R^2}{d_1^2}\right) - 2\frac{R}{d_1}\cos\theta\right]^{3/2}} \left(\frac{R}{d_1}\right) \cdot \left(1 - \frac{R^2}{d_1^2}\right)$$

$$= -\frac{q}{2} \cdot \frac{R}{d_1}\left(1 - \frac{R^2}{d_1^2}\right) \int_0^{\pi} d\theta \, \frac{\sin\theta}{\left[1 + \frac{R^2}{d_1^2} - 2\frac{R}{d_1}\cos\theta\right]^{3/2}}$$

$$= +\frac{q}{2} \frac{\cancel{R}}{\cancel{d_1}}\left(1 - \frac{R^2}{d_1^2}\right) \cdot \left[\frac{\cancel{d_1}}{\cancel{R}} \cdot \left(1 + \frac{R^2}{d_1^2} - 2\frac{R}{d_1}\cos\theta\right)^{-1/2}\right]\Bigg|_{\theta=0}^{\theta=\pi}$$

$$= +\frac{q}{2}\left(1 - \frac{R^2}{d_1^2}\right)\left[\frac{1}{\left[1 + \frac{R^2}{d_1^2} + 2\frac{R}{d_1}\right]^{1/2}} - \frac{1}{\left[1 + \frac{R^2}{d_1^2} - 2\frac{R}{d_1}\right]^{1/2}}\right]$$

$$= +\frac{q}{2} \cdot \left(1 + \frac{R}{d_1}\right)\left(1 - \frac{R}{d_1}\right)\left[\frac{1}{\left(1 + \frac{R}{d_1}\right)} - \frac{1}{\left(1 - \frac{R}{d_1}\right)}\right]$$

$$= +\frac{q}{2} \cdot \left[\left(1 - \frac{R}{d_1}\right) - \left(1 + \frac{R}{d_1}\right)\right]$$

$$= +\frac{q}{2} \cdot \left(-2\frac{R}{d_1}\right) = -q\left(\frac{R}{d_1}\right) = q' \quad \checkmark$$