PHY 611 – Electromagnetic Theory I
Problem Set 1
Due: Wednesday, August 29 at 10:00 a.m. at the start of class

This first problem set will review a number of different mathematical techniques which we will use frequently the rest of the semester.

Problem 1: Cylindrical Coordinates [5 points]

(a) Consider the cylindrical coordinate system, (ρ, ϕ, z). Derive expressions for the cylindrical unit vectors, $(\hat{\rho}, \hat{\phi}, \hat{z})$, in terms of the Cartesian unit vectors, $(\hat{x}, \hat{y}, \hat{z})$. Then, using these results, show that

$$\frac{\partial \hat{\rho}}{\partial \phi} = \hat{\phi}, \qquad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\rho}.$$

(b) Consider a vector written in cylindrical coordinates, $\vec{V} = V_\rho \hat{\rho} + V_\phi \hat{\phi} + V_z \hat{z}$. The gradient operator in cylindrical coordinates is

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}.$$

Show, by explicitly calculating the dot product of $\vec{\nabla}$ with $\vec{V}$, that the divergence of $\vec{V}$ is

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}.$$

Problem 2: Spherical Coordinates [5 points]

(a) Consider the spherical coordinate system, (r, θ, ϕ). Derive the following expressions relating the spherical unit vectors to the Cartesian unit vectors

$$\begin{aligned}
\hat{x} &= \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi, \\
\hat{y} &= \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi, \\
\hat{z} &= \hat{r}\cos\theta - \hat{\theta}\sin\theta.
\end{aligned}$$

(b) Derive expressions for $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ in terms of $\partial/\partial r$, $\partial/\partial \theta$, and $\partial/\partial \phi$. *Hint:* Equate $\vec{\nabla}_{xyz}$ with $\vec{\nabla}_{r\theta\phi}$.

Problem 3: Index Notation [10 points]

Show, using index notation, that

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A}) + (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B}.$$

You will probably find the following identity useful: $\epsilon_{kij}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{\ell j}$.

Problem 4: Vector Calculus and Differentials [15 points]

Suppose $\vec{\nabla} \cdot \vec{B} = 0$ for $|\vec{x}| < R$. Show that one solution of $\vec{\nabla} \times \vec{A} = \vec{B}$ is

$$\vec{A}(\vec{x}) = -\int_0^1 dt\; t\; \vec{x} \times \vec{B}(t\vec{x}).$$

Hint: Calculate $\vec{\nabla} \times \vec{A}$. Then, let $\vec{y} = t\vec{x}$ and calculate $\frac{d}{dt}\vec{B}(t\vec{x})$ in terms of $\vec{\nabla}_{\vec{y}}$ acting on $\vec{B}(\vec{y})$.

Problem 5: Dirac Delta Function [10 points]
Suppose we define a sequence $\delta_n(x) = n/(2\cosh^2 nx)$.

(a) Show that $\forall\ n$

$$\int_{-\infty}^{\infty} dx\ \delta_n(x) = 1.$$

(b) Show that

$$\int_{-\infty}^{x} dx'\ \delta_n(x') = \frac{1}{2}\left[1 + \tanh nx\right] \equiv u_n(x),$$

where

$$\lim_{n\to\infty} u_n(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases}$$

which is a representation of the Heaviside step function.

(c) Prove the identity

$$\delta\Big(g(x)\Big) = \sum_{a,\ g(a)=0,\ g'(a)\neq 0} \frac{\delta(x-a)}{|g'(a)|}.$$

Hint: Decompose the integral $\int_{-\infty}^{\infty} dx\ f(x)\ \delta(g(x))$ into a sum of integrals over small intervals containing the zeros of $g(x)$.

Problem 6: Hyperbolic Trigonometry [5 points]
Show that

$$\sinh^2 x + \sinh^2 y = \sinh^2(x-y) + 2\cosh(x-y)\sinh x \sinh y.$$

Problem 7: Integrals [20 points]
Work the following problems *by hand*. You must show sufficient work such that I am convinced you did not use Mathematica, Maple, etc.

(a) Show that for a real-valued β and $\beta > 0$

$$\int_{-\infty}^{\infty} \exp\left[-\beta(x+iy)^2\right] dx = \sqrt{\pi/\beta},$$

where, as usual, $i = \sqrt{-1}$.

(b) Consider the following integral (which, actually, represents a wave packet in one-dimensional quantum mechanics)

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \exp\left[-\frac{(k-k_0)^2}{4\sigma^2}\right] \exp\left[-ikx_0\right] \exp\left[i\left(kx - \frac{\hbar}{2m}k^2 t\right)\right].$$

Show that $\psi(x,t)$ can be rewritten as

$$\psi(x,t) = \frac{\left(2\sigma^2/\pi\right)^{1/4}}{\sqrt{\left(1 + \frac{2i\hbar\sigma^2 t}{m}\right)}} \exp\left[\frac{-\sigma^2(x-x_0)^2 + ik_0(x-x_0) - \frac{i\hbar k_0^2 t}{2m}}{1 + \frac{2i\hbar\sigma^2 t}{m}}\right].$$

(c) Now show explictly by performing the integral that

$$[\Delta x(t)]^2 = [\Delta x(0)]^2 + \left(\frac{\hbar t}{2m\Delta x(0)}\right)^2,$$

where

$$[\Delta x(t)]^2 \equiv \int_{-\infty}^{\infty} dx\ |\psi(x,t)|^2 [x - (x_0 + \hbar k t/m)]^2.$$

Problem 8: Fourier Series [10 points]
Consider the expansion of a function $f(x)$ on the interval $[0, 2\pi]$ in terms of a Fourier series

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + \sum_{n=1}^{\infty} B_n \sin nx,$$

with the coefficients A_0, A_n, and B_n related to the function $f(x)$ by

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^{2\pi} dx\ f(x)\cos nx, \\ B_n &= \frac{1}{\pi}\int_0^{2\pi} dx\ f(x)\sin nx, \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{1}$$

(a) Suppose, instead, that $f(x)$ is to be represented by (or fitted to) a *finite* Fourier series (i.e., the sum is to be truncated at some $n = N$, as is required for any finite amount of CPU time). A measure of the accuracy of such a finite series is given by the integrated square of the deviation,

$$\Delta_N \equiv \int_0^{2\pi} dx \left[f(x) - \frac{1}{2}A_0 - \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \right]^2.$$

Show that the requirement that Δ_N be minimized leads to the same choice of coefficients as above (i.e., they do not depend on N!).

(b) Consider the representation of a triangular wave on the interval $[-\pi, \pi]$ (note the change in the interval)

$$f(x) = \begin{cases} -x, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$$

Find a Fourier series representation for $f(x)$.

Problem 9: Complex Variables [10 points]
Let $z = x + iy$ be a complex number.

(a) Prove the triangle inequality for complex numbers z_1 and z_2,

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

Interpret this result in terms of two-dimensional vectors in the (x, y) plane.

(b) Show that

$$\arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2,$$

where ‘arg’ denotes the ‘argument’, or phase, of a complex number.

(c) Power series expansions for the elementary functions, such as sine, cosine, etc., can be defined in the complex plane. For example, the usual power series expansions for sine and cosine hold,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Using these, derive the following identities

$$\begin{aligned}
\sin(x+iy) &= \sin x \cosh y + i \cos x \sinh y, \\
\cos(x+iy) &= \cos x \cosh y - i \sin x \sinh y, \\
|\sin z|^2 &= \sin^2 x + \sinh^2 y, \\
|\cos z|^2 &= \cos^2 x + \sinh^2 y.
\end{aligned}$$

Thus, we see that we can have $|\sin z|$, $|\cos z| > 1$ in the complex plane!

Problem 10: Taylor Expansions [10 points]

(a) Consider a function of a single variable, $f(x)$. Suppose we know the value of $f(x)$ at discrete grid points along the x-axis, $x = 0$, $\pm h$, $\pm 2h$, etc, all separated by a "step size" h. A "forward difference approximation" to the value of the first derivative, $f'(x)$, at some point x is

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

whereas a "centered difference approximation" to $f'(x)$ is

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

Which of these (if either) provides a better (i.e., more accurate) approximation to $f'(x)$?

(b) Consider the same function $f(x)$ as in part (a). Show that a "centered difference approximation" to the second derivative $f''(x)$ is

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

(c) Now consider a function in three-dimensional space, $\Phi(x, y, z)$. Again, we will assume we know the value of Φ on a three-dimensional grid of discrete (x, y, z) points, all separated by the same step size h in all three dimensions. Label these grid points with the indices (i, j, k). Suppose Φ satisfies the Laplace Equation, $\vec{\nabla}^2\Phi(x, y, z) = 0$. Show that the Laplace Equation can be "discretized" at any point (i, j, k) into the form

$$\Phi(i{+}1, j, k) + \Phi(i{-}1, j, k) + \Phi(i, j{+}1, k) + \Phi(i, j{-}1, k) + \Phi(i, j, k{+}1) + \Phi(i, j, k{-}1) - 6\Phi(i, j, k) = 0.$$

Such discretization forms the basis of elementary "relaxation" methods for the numerical solution of Laplace's Equation. We will explore this technique later for the numerical solution of boundary value problems.