PHY 611 – Electromagnetic Theory I
Problem Set 2
Due: Wednesday, September 12 at 10:00 a.m. at the start of class

Problem 1 [5 points]: Coulomb's Law
Consider a thin, uniformly charged rod with total charge Q and length $2d$ oriented along the z-axis, with its center located at $z = z_0$. Then consider a thin, uniformly charged ring of radius a and total charge Q' oriented in the xy-plane and centered on the z-axis. Calculate the Coulomb force between the ring and the rod (via any method of your choosing), and show that your expression for the force reduces to the expected form in the limit that $z_0/d \gg 1$.

Problem 2 [5 points]: Electric Field
Calculate the electric field at a height h above the center of a finite square sheet (of size $a \times a$) with a uniform surface charge density of σ. Show that your result converges to the appropriate forms in the limiting cases of $a/h \gg 1$ and $a/h \ll 1$.

Problem 3 [5 points]: Electrostatic Potential
Consider two infinitely long, parallel line charges with equal and opposite linear charge densities, $\pm\lambda$, where λ has units of charge/length. Show that the equipotentials are infinitely long cylinders.

Problem 4 [15 points]: Inverse Square Law and the Electrostatic Potential
Suppose that electromagnetism does not obey a true inverse square law[1], such that the electric field of a point charge is $\vec{E} \propto r^{-(2+\delta)}\hat{r}$, where $|\delta| \ll 1$.

(a) Calculate $\vec{\nabla} \cdot \vec{E}$ and $\vec{\nabla} \times \vec{E}$ for $r \neq 0$. Find the electric potential for a point charge.

(b) Now suppose two concentric spherical conducting shells of radii a and b, where $a > b$, are joined by a thin conducting wire. Show that if charge Q_a resides on the outer shell of radius a, then the charge on the inner shell of radius b is

$$Q_b \approx -\frac{Q_a \delta}{2(a-b)} \left[2b \ln 2a - (a+b)\ln(a+b) + (a-b)\ln(a-b)\right].$$

Thus, a measurement of the ratio Q_b/Q_a would, in principle, provide for a stringent experimental test of the accuracy of the inverse square law of electromagnetism.

Problem 5 [15 points]: Electrostatic Potential of a Uniform Dipole Layer
Calculate the potential $\Phi(z)$ along the axis of a disk of radius R in two cases:

(a) If the disk is covered with a uniform charge density σ.

(b) If the disk is covered with a uniform dipole layer of dipole moment density $\vec{D} = D\hat{z}$ per unit area (assume $D > 0$). Begin by assuming that this uniform dipole layer is actually composed of a system of two plates of equal and opposite charge densities $\pm\sigma$ located at $z = \pm d/2$, with $D \equiv \sigma d$ finite in the limit $d \to 0$. Show that in the limit of $z \gg R \gg d$ your result gives the correct limiting form for a point dipole. Then, find the value of $\Phi(z = 0^+) - \Phi(z = 0^-)$ in the limit of $d \to 0$. Interpret your result for $\Phi(z = 0^+) - \Phi(z = 0^-)$ in terms of the electric field between the two plates.

[1] In fact, there has been significant experimental work to test the validity of the inverse square laws of both electromagnetism and gravity. The inverse square law for gravity has been shown to be valid down to length scales of $\sim 60\ \mu$m, placing stringent constraints on the "sizes" of any extra dimensions, such as those arising in string theory. See, for example, D. J. Kapner *et al.*, Phys. Rev. Lett. **98**, 021101 (2007).

Problem 6 [35 points]: Green's Theorem in Two Dimensions

In this problem we will develop a two dimensional version of Green's Theorem. In the next problem set we will then use this two dimensional version of Green's theorem to solve boundary value problems.

(a) We begin by considering a two dimensional domain D. We want to find the simplest function (the so-called "fundamental solution") which satisfies $\vec{\nabla}'^2 v(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}')$, where the Laplacian is with respect to $\vec{x}'$. Solve the equation $\vec{\nabla}'^2 v(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}')$. *Hint:* You may find the following useful. In two dimensions, given a vector field $\vec{F}$ defined over a domain D with a closed boundary C, the two dimensional version of the divergence theorem is

$$\iint_D \vec{\nabla}\cdot\vec{F}\ da = \int_C \vec{F}\cdot\hat{n}\ d\ell,$$

where $\hat{n}$ is normal to C. *Note:* Just as in three dimensions, your solution for $v(\vec{x},\vec{x}')$ is the "fundamental solution", not the Green's function in two dimensions (which is geometry dependent). The two dimensional Green's function will be $G(\vec{x},\vec{x}') = v(\vec{x},\vec{x}')+h(\vec{x},\vec{x}')$, where $h(\vec{x},\vec{x}')$ is constructed to simultaneously satisfy the Laplace equation inside of the domain and the appropriate boundary condition (Dirichlet or Neumann).

(b) Now let $\vec{V} = v\vec{\nabla}u$, where u and v are now arbitrary scalar functions. Show that the two dimensional version of Green's first identity is

$$\iint_D v\vec{\nabla}^2 u\ da = -\iint_D \vec{\nabla}u\cdot\vec{\nabla}v\ da + \int_C v\frac{\partial u}{\partial n}\ d\ell.$$

(c) Show that the two dimensional version of Green's second identity is

$$\iint_D (v\vec{\nabla}^2 u - u\vec{\nabla}^2 v)\ da = \int_C \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) d\ell.$$

(d) Now let $v = \ln r$ (does this look familiar?), where $r = \sqrt{(x-x')^2+(y-y')^2}$ for $\vec{x}=(x,y)$ is interior to the closed boundary C. Show that

$$2\pi u(\vec{x}) = \iint_D \ln r\ \vec{\nabla}'^2 u(\vec{x}')\ da' - \int_C \left(\ln r\frac{\partial u(\vec{x}')}{\partial n'} - u(\vec{x}')\frac{\partial \ln r}{\partial n'}\right)\ d\ell'.$$

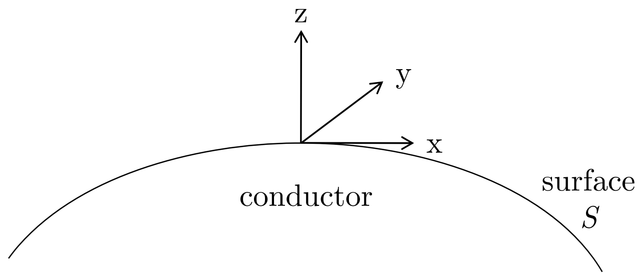
Just as in three dimensions, we can then identify $u(\vec{x})$ with $\Phi(\vec{x})$, and by "replacing" the fundamental solution v with the Green's function, we will thus have a two dimensional integral equation for $\Phi(\vec{x})$.

Problem 7 [20 points]: Laplace Equation

Let S be the surface of a charged conductor with no nearby charges (see diagram below and the corresponding coordinate system). Suppose we choose a coordinate system such that $x=y=z=0$ is a point on S, and suppose further that the equation for S near the origin can be written as

$$z = -\frac{x^2}{2R_1} - \frac{y^2}{2R_2},$$

where $R_{1,2}$ are constants. Calculate the value of $\dfrac{1}{E_z}\dfrac{\partial E_z}{\partial z}$ at $x=y=z=0$.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