PHY 611 – Electromagnetic Theory I
Problem Set 3
Due: Wednesday, September 26 at 10:00 a.m. at the start of class

Problem 1 [10 points]: "Self Energy" Contributions to the Electrostatic Energy

(a) Read the discussion on pp. 41–42 of Jackson on the contributions of the "self energy" terms to the electrostatic energy density. First, derive Eq. (1.58) from Eq. (1.57). Second, show, as stated after Eq. (1.58), that "...the dimensionless integral can easily be shown to have the value 4π, so that the interaction energy reduces to the expected value".

(b) Consider two arbitrary charge distributions. Show that the sum of the "self energies" of the two charge distributions is always greater than or equal to their interaction energy.

Problem 2 [5 points]: Capacitance of Parallel Cylinders

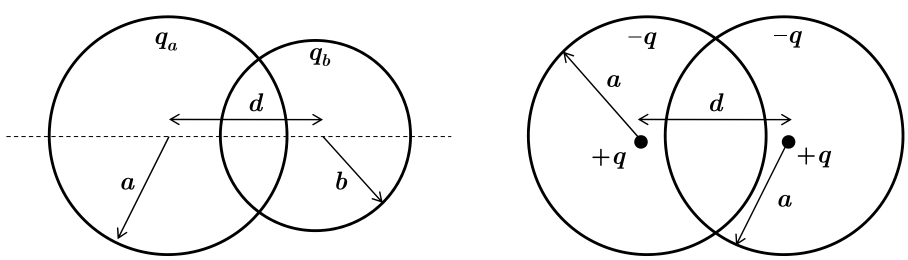
Neglecting end effects, calculate the capacitance per unit length, C', of a system of two long parallel cylinders of radius a whose axes are separated by a distance $d \gg a$.

Problem 3 [20 points]: Electrostatic Interaction Energy of Two Atoms

In this problem we will consider a simple model for the electrostatic interaction energy of two atoms. (Nevermind how either of the configurations shown below could actually exist; the result is, nevertheless, quite realistic!) We will not consider any electrostatic "self energies" in this problem.

(a) Calculate the electrostatic interaction energy W for the configuration shown below on the left, consisting of two (negligibly thin) interpenetrating, spherical shells of radii a and b. The shells have uniformly distributed surface charges of q_a and q_b, and are separated by a distance d, where $a - b < d < a + b$.

(b) As shown below on the right, a highly idealized electrostatic model of an atom consists of a point charge $+q$ (i.e., the "nucleus") located at the center of a (negligibly thin) spherical shell with a uniformly distributed charge of $-q$ (i.e., the "electron shell"). Working within this model, calculate the electrostatic interaction energy W of two identical atoms with radii a separated by an internuclear separation d for three cases: $d < a$; $a < d < 2a$; and $d > 2a$. Make a qualitative sketch of W as a function of d over this same range of d. If there is a minimum in $W(d)$, calculate the value of d at which this occurs in terms of the given parameters.



Problem 4 [20 points]: Method of Images

Consider two conducting spheres, of radii R_1 and R_2, where in general we will assume $R_1 \neq R_2$. Suppose the spheres are in contact at a single point. For example, if the sphere of radius R_1 is centered at $(0, 0, 0)$, the other sphere would be centered at $(R_1 + R_2, 0, 0)$. Suppose that there are charges, Q_1 and Q_2, on the surfaces of the spheres such that they are at the same potential $V_0 \neq 0$ (enforced by the fact that they are in contact).

(a) Find recursion relations which would permit you to calculate Q_1 and Q_2.

(b) Consider a case where $R_1 = 1.0$ m, and $R_2 = 0.0001$ m. Write a computer code to calculate the numerical value of the ratio Q_2/Q_1 for this case. Make sure you iterate your recursion to a sufficiently high order so that your value for Q_2/Q_1 is stable. You can feel free to use any language you wish, but you will need to include a print-out of your code. *Note: This will be a nice illustration that the charges on the two spheres are not equal, even though they are at the same potential!*

Problem 5 [10 points]: Dirichlet Green Function for the Sphere
A conducting sphere of radius R is centered on the origin, and is held at a potential of $V \neq 0$. Two point charges, both of identical charge $Q > 0$, are located at positions $(0, 0, a)$ and $(0, 0, -b)$, where $a > R$ and $b > R$ (i.e., both charges are outside the sphere). Take $\Phi(\infty) = 0$.

(a) Using the Green function technique, solve for the potential $\Phi(z)$ along the z-axis in the region external to the sphere (i.e., for $r > R$).

(b) Determine the value for the z-component of the electric field at $z = R^+$ (i.e., just above the "North Pole" of the sphere).

Problem 6 [20 points]: Dirichlet Green Function in Two Dimensions for a Circle
Consider a circle of radius a. Use polar coordinates (ρ', ϕ'). Suppose the boundary condition is such that the potential on the circle is specified to be some function $f(\rho' = a, \phi')$. Using your results from the previous problem set for Green's Theorem in two dimensions, show that at any point (ρ, ϕ) exterior to the circle, the potential is

$$\Phi(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} d\phi' \, f(a, \phi') \, \frac{\rho^2 - a^2}{\rho^2 + a^2 - 2a\rho\cos(\phi - \phi')}.$$

This is termed the "Poisson integral formula" for the exterior problem for a circle. *Hint:* You will need to find the functional form of the Dirichlet Green function for a circle in two dimensions. Think of how we came to find the functional form for the Dirichlet Green function for a sphere in three dimensions.

Problem 7 [15 points]: Neumann Green Function in Two Dimensions
We now want to consider the Green function for the Neumann boundary condition in two dimensions (i.e., solution for the potential when the normal derivative of the potential, $\partial\Phi/\partial n'$, is specified on the boundary).

(a) In class, we showed that the simplest allowable boundary condition on a Neumann Green function, $G_N(\vec{x}, \vec{x}')$, in three dimensions is

$$\frac{\partial G_N}{\partial n'}(\vec{x}, \vec{x}') = -\frac{4\pi}{S},$$

where S denotes the total surface area of the surface bounding the volume V. Starting from the two dimensional version of Green's Theorem which you derived in the previous problem set, show that the simplest allowable boundary condition on a Neumann Green function in two dimensions is

$$\frac{\partial G_N}{\partial n'}(\vec{x}, \vec{x}') = C,$$

where C is some constant. What is the physical interpretation of C?

(b) If we were attempting to solve a two dimensional potential problem in the upper half plane (i.e., the $y > 0$ portion of the xy-plane, with the boundary being the entire x-axis, from $x = -\infty$ to $x = +\infty$), is it permissible to specify $\partial G_N(\vec{x}, \vec{x}')/\partial n' = 0$ on the boundary?

(c) The functional form for Neumann Green functions in two dimensions can be constructed by considering a "positive image charge" (whereas, as you recall, we constructed Dirichlet Green functions in three dimensions by considering a "negative image charge", which produced the result that the Dirichlet Green function was zero everywhere on the surface). For the case of the upper half plane discussed in part (b), write a functional form for the Neumann Green function, and show that it satisfies the permissible condition on $\partial G_N(\vec{x}, \vec{x}')/\partial n'$ on the boundary (i.e., the x-axis) and also $\vec{\nabla}'^2 G_N(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}')$.