PHY 611 – Electromagnetic Theory I
Problem Set 4
Due: Wednesday, October 10 at 10:00 a.m. at the start of class

References Policy: On the first page of your submitted work, please list all of your collaborators and any references/sources that you consulted (e.g., textbooks, journal articles, web pages, etc.).

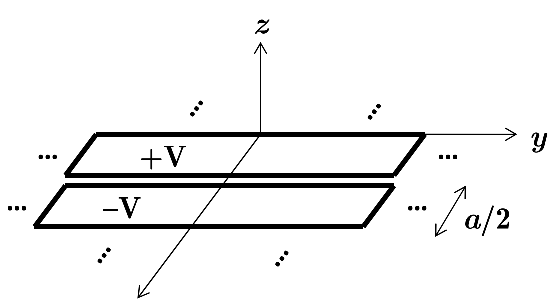
Problem 1 [10 points]: Poisson (not Laplace) Equation
Before the era of transistors, a common device was a vacuum diode. A vacuum diode is a parallel plate capacitor with a potential difference V across a gap of width d, all of which is enclosed within a vacuum tube. The cathode (at $\Phi = 0$) is heated, so electrons which are "boiled off" flow to the anode (at $\Phi = V$). The cloud of moving electrons within the gap (called "space charge") quickly builds up to the point where it reduces the electric field at the surface of the cathode to zero. From then on, a steady current flows between the plates.

Consider the steady-state situation in which a *constant* current density $j = \rho(x)v(x)$ flows, and assume the electrons leave the cathode with velocity $v(0) = 0$. Here, $\rho(x)$, for $0 \leq x \leq d$, is the electron charge density.

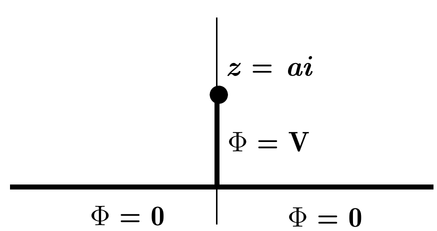
(a) Solve for the potential $\Phi(x)$ in the region $0 \leq x \leq d$ via the Poisson equation, $\vec{\nabla}^2\Phi(x) = -\rho/\epsilon_0$. How does your result for the potential compare to an ordinary capacitor?

(b) Solve for the charge density $\rho(x)$ and the current density j.

Problem 2 [10 points]: Separation of Variables in Rectangular Coordinates
Consider an array of thin conducting strips lying in the $z = 0$ plane. Each strip is $a/2$ wide in the x-direction, and infinitely long in the y-direction. The strips are alternately at potentials of $\Phi = +V$ and $\Phi = -V$, and insulated from each other by negligibly thin insulators. Derive an expression for the potential $\Phi(x, z)$ in the upper half-space $z > 0$. Make a (computer-generated) plot of $\Phi(x, z)/V$ for $z = 0$ as a function of the scaled coordinate x/a.



Problem 3 [10 points]: Conformal Mapping
Find the electrostatic potential Φ in the upper half-plane with the boundary values shown below.

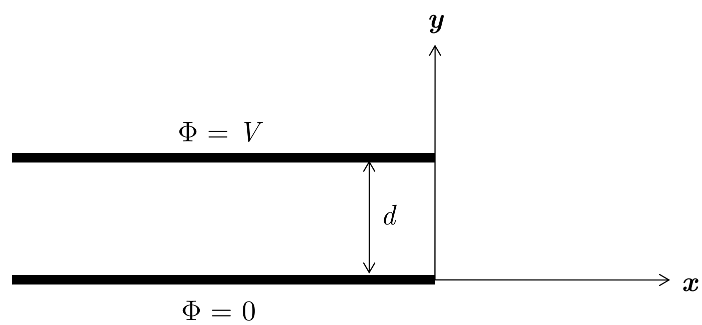


Problem 4 [20 points]: Conformal Mapping
Read Section 2.10 of Jackson (you can also refer to my lecture notes, which filled in many of the missing details). As you recall, we solved this problem in class directly via separation of variables. Now, instead, solve this same potential problem via an appropriate Schwarz transformation. Note that you may need to do this transformation in several steps. Show that your answer is consistent with Eq. (2.65) of Jackson.

Problem 5 [30 points]: Fringe Fields in a Finite-Sized Capacitor
As you know, the capacitance of two large conducting sheets of area A which are separated by a distance d is $C = \epsilon_0 A/d$. Recall that this result can be derived assuming that the electric field is perfectly uniform everywhere in the region between the plates and is zero outside of the region between the plates (i.e., the so-called "fringe fields", or "end effects", are neglected).

In this problem we will explore the impact of such fringe fields on the capacitance of a system composed of *finite* sized conducting plates. As a starting point, consider two semi-infinite parallel plates, separated by a distance d, as shown below. Using an appropriate Schwarz transformation, find an expression for the total charge per unit length perpendicular to the xy-plane between $x = 0$ and $x = x_0$ (where $x_0 < 0$) under the approximation $|x_0| \gg d$. You should be able to write an expression for the total charge which is the sum of two terms: one if there were no fringe effects (i.e., for plates of infinite extent), and another which is the correction due to the fringe fields.

Then, apply your result to the problem of a parallel plate capacitor composed of plates of area $A = ab$ separated by a distance d. Again, assuming $a \gg d$ and $b \gg d$, find an expression for the capacitance accounting for fringe effects.



Problem 6 [20 points]: Slightly Non-Concentric Spherical Capacitor
A spherical capacitor consists of two conducting spherical shells of radii a and b, where $a < b$. However, their centers are displaced by a small amount $c \ll a$. Take the center of the sphere with radius a as the origin, and also take $\hat{z}$ along the line connecting the centers of the spheres.

(a) Show that the equation for the surface of the sphere of radius b in spherical coordinates is

$$r = b + cP_1(\cos\theta) + \mathcal{O}(c^2).$$

(b) Suppose the sphere with radius a is grounded and the sphere with radius b is at a potential V. Show that the potential in the region between the spheres is given by

$$\Phi(r,\theta) = V\left[\frac{r-a}{b-a}\left(\frac{b}{r}\right) - \frac{abc}{r^2(b-a)}\left(\frac{r^3-a^3}{b^3-a^3}\right)P_1(\cos\theta) + \mathcal{O}(c^2)\right].$$

(c) What is the capacitance, to $\mathcal{O}(c)$?

Bonus Problem [5 Midterm Exam points]: Numerical Solution to the Laplace Equation In class we discussed the iterative Jacobi method (i.e., "relaxation") for the numerical solution to the Laplace Equation in the presence of boundary conditions. In this optional bonus problem, we will apply this method to the numerical solution of the boundary value problem discussed in Section 2.9 of Jackson. For numerical concreteness, set $a = b = c = 1$. Set the potential on the top face of the cube to be $V(x, y) = 1$, with the potential on the other five faces of the cube set to zero.

Divide the cube into an $11 \times 11 \times 11$ grid, such that the spacing between grid points is $\Delta x = \Delta y = \Delta z = 0.1$. Write a computer code to perform Jacobi iteration over all of the grid points. Iterate your algorithm for a sufficiently large number of iterations such that your results have converged (i.e., are numerically stable). Make a plot of your calculated results for the potential as a function of z for $(x, y) = (0.5, 0.5)$ [i.e., of the values of the potential at the 11 grid points between $z = 0$ and $z = 1$ for $(x, y) = (0.5, 0.5)$] and compare your results with the exact expression obtained via separation of variables, Eqs. (2.57) and (2.58) in Jackson.

To receive credit for this problem, you must submit a plot of the requested results, and also an electronic copy of your code via email.