PHY 611 – Electromagnetic Theory I
Problem Set 6
Due: Monday, November 19 at 10:00 a.m. at the start of class

References Policy: On the first page of your submitted work, please list all of your collaborators and any references/sources that you consulted (e.g., textbooks, journal articles, web pages, etc.).

Problem 1 [20 points]: Charged Particle Transport in Electric and Magnetic Fields
This is somewhat of an open-ended question, kind of like one you might encounter in the context of a research project. I want you to think about the physics, and then write-up what you think will happen. Consider a particle with charge q moving with some constant velocity $\vec{v}$ in uniform electric $\vec{E}$ and magnetic $\vec{B}$ fields, which are neither perfectly parallel or perpendicular. Thus, you can write $\vec{E}$ in terms of non-zero components parallel and perpendicular to $\vec{B}$, and $\vec{E}\times\vec{B}\neq 0$. Describe (via words, equations, sketches, whatever, ...) the resulting trajectory of the charged particle. You should treat all motion and fields non-relativistically.

Problem 2 [15 points]: Nima's Spherical Coil
A total charge Q is distributed uniformly over the surface of a sphere of radius R. Suppose the sphere starts to rotate (carrying the charge with it) at an angular velocity ω. Calculate the resulting $\vec{B}$ field everywhere. It is probably easiest to do this problem by first writing an expression for the surface current that develops on the surface of the sphere, and then proceeding to calculate the vector potential. [FYI: Nima is an expert on this problem!]

Problem 3 [20 points]: Vector Potential of a Circular Current Loop
[*Part of this problem is written up in Jackson pp. 181–183, but in spherical coordinates. The point is for you to acquire some practice with these techniques.*] A circular loop of wire of radius a and negligible cross section carrying a steady current I lies in the xy-plane with its center at the origin. A point in space is indicated by cylindrical coordinates (ρ, ϕ, z).

(a) Starting from the definition of the vector potential [i.e., Eq. (5.32) in Jackson] show that for this circular current loop the vector potential at any point (ρ, ϕ, z) is of the form

$$\begin{aligned}\vec{A} &= \frac{\mu_0 I a}{2\pi}\int_0^{\pi} d\phi \frac{\cos\phi}{(a^2+\rho^2+z^2-2a\rho\cos\phi)^{1/2}}\hat{\phi}, \\ &= \frac{\mu_0 I}{\pi k}\left(\frac{a}{\rho}\right)^{1/2}\left[\left(1-\frac{1}{2}k^2\right)K(k)-E(k)\right]\hat{\phi},\end{aligned}$$

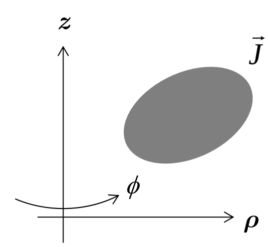
where $k^2 = 4a\rho/[(a+\rho)^2+z^2]$, and $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind.[1]

(b) Instead of writing the above integral expression for $\vec{A}$ in terms of elliptic integrals, instead expand the integrand and show that in the limit of $(\rho^2+z^2)^{1/2} \gg a$ you obtain the expression for the vector potential of a magnetic dipole, and then the correct expression for the $\vec{B}$ field of a magnetic dipole. In calculating the $\vec{B}$ field, you might find it easier to then work in spherical coordinates.

[1] You can find the definitions of $K(k)$ and $E(k)$ on Wikipedia. Note that in other references these might instead be defined to be $K(k^2)$ and $E(k^2)$ (e.g., as in Arfken). You might find it helpful to change variables to $\alpha = \pi - \phi$ at some point.

Problem 4 [15 points]: Magnetic Dipole Moment of a Rotating Sphere
A sphere of radius a has a charge $+Q$ distributed uniformly over its surface, and a charge $-Q$ distributed uniformly throughout its volume. The sphere rotates with angular velocity ω. Calculate the magnetic dipole moment of the sphere.

Problem 5 [15 points]: Magnetic Field of a Cylindrically Symmetric Current Density
A cylindrically symmetric current density $\vec{J}$ flows in the ρ-z plane, as indicated in the figure at the top of the next page. The current density is cylindrically symmetric, meaning if we write $\vec{J} = J_\rho \hat{\rho} + J_\phi \hat{\phi} + J_z \hat{z}$, $J_\phi = 0$, and J_ρ and J_z are functions only of ρ and z. The current density $\vec{J}$ is confined to the gray shaded region, which is symmetric in ϕ about the z-axis, and zero everywhere outside of the shaded region. Prove that $\vec{B}$ is zero outside the shaded region.



Problem 6 [15 points]: Helmholtz Coils
Suppose your advisor wants you to build a system of magnetic coils which will generate a highly-uniform magnetic field. You then proceed to do so by constructing a pair of so-called "Helmholtz coils", consisting of two circular coil windings of radius a, each carrying the same current I (or, for the experimentalists, the same number of "Amp-turns" wound in the same direction). You then setup your coils in your laboratory so that their centers are at $z = \pm b$.

(a) Calculate the coil separation distance $2b$ such that the first, second, and third derivatives of $|\vec{B}(z)|$ along the coil's axes vanish at $z = 0$. Thus, the magnetic field is very uniform at the center of a pair of Helmholtz coils. This separation distance is termed the "Helmholtz condition".

(b) Now suppose your advisor is not satisfied with the uniformity of the magnetic field generated by your pair of Helmholtz coils. S/he then suggests you add a second pair of Helmholtz coils of radius $a' = a/2$, positioned such that they satisfy their own "Helmholtz condition". What current I' should flow in this second pair so as to cancel the fourth derivative of $|\vec{B}(z)|$ of the first pair? What fraction of the original central field magnitude is "lost" in this configuration?