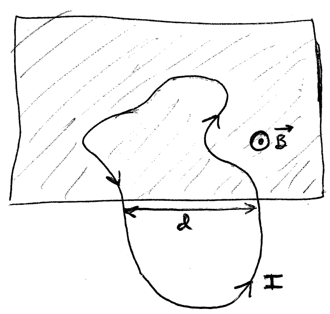
PHY 611 – Electromagnetic Theory I
Problem Set 7
Due: Monday, December 3 at 10:00 a.m. at the start of class

References Policy: On the first page of your submitted work, please list all of your collaborators and any references/sources that you consulted (e.g., textbooks, journal articles, web pages, etc.).

Problem 1 [10 points]: Magnetic Force on a Current Loop
A loop of wire bent into some arbitrary shape is confined to the xy-plane. As shown below, part of the loop is in a uniform magnetic field $\vec{B} = B\hat{z}$ (indicated by the shaded region); everywhere else the magnetic field is zero. If the current in the loop is I, find an expression for the magnetic force $\vec{F}$ on the loop in terms of I, B, and the distance d indicated on the figure.



Problem 2 [20 points]: Boundary Conditions for a Cylinder

(a) Before working parts (b) and (c), first derive the functional form of the solution (via separation of variables) to the Laplace Equation $\vec{\nabla}^2\Phi = 0$ in two dimensions in cylindrical/polar coordinates (ρ, ϕ).

Now, consider a long (solid) cylinder of radius R composed of material with permeability $\mu > \mu_0$ which is oriented along the z-axis. The cylinder is placed in a field $\vec{H} = H_0\hat{x}$. Calculate the $\vec{B}$ field inside the cylinder:

(b) by solving $\vec{H} = -\vec{\nabla}\Phi_M$; and

(c) by solving $\vec{B} = \vec{\nabla} \times \vec{A}$, where $\vec{A} = a(\rho, \phi)\hat{z}$ in cylindrical coordinates (ρ, ϕ, z).

Problem 3 [20 points]: Magnetic Shielding Factor of a Cylindrical Shell
My former graduate student Susan Malkowski worked quite a bit over the years on the development of various types of "magnetic shields". Here, we will calculate the "shielding factor" of a cylindrical magnetic shell. Consider our "magnetic shield" to consist of a thin cylindrical shell composed of material with permeability $\mu > \mu_0$. Let t denote the thickness of the material, and let R denote the radius of the cylindrical shell (you can take R be the average of the cylindrical shell's "inner" and "outer" radius). Assume the cylinder is "long", with its axis oriented along the z-axis, and assume the cylinder is placed in some constant magnetic field $\vec{H}_0 = H_0\hat{x}$.

(a) Show that the ratio of $|\vec{H}|$ in the region interior to the cylindrical shell to $|\vec{H}_0|$ is (assuming $R \gg t$)

$$\frac{|\vec{H}_0|}{|\vec{H}|} = 1 + \frac{(\mu/\mu_0)t}{2R}.$$

This ratio is termed the "shielding factor". Thus, cylindrical shells with large μ/μ_0 are quite effective at "shielding" their interior regions from external fields.

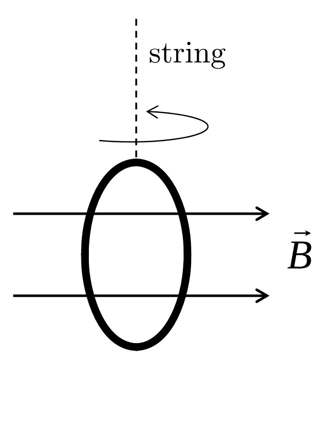
(b) As reported in an article authored by my research group [S. Malkowski *et al.*, Review of Scientific Instruments **82**, 075104 (2011)] the permeability of a magnetic material called "Metglas" is $\mu/\mu_0 \sim 5 \times 10^6$. Assume you construct a cylindrical magnetic shield by placing a thin, 100 μm-thick layer of Metglas onto the surface of a 0.5-m diameter cylindrical shell. Suppose you then orient the axis of this shield perpendicular to the direction of the Earth's magnetic field, which you can take to have a magnitude of 0.5 Gauss $= 5 \times 10^{-5}$ Tesla. What would you expect the magnitude of the shielded field in the region interior to the cylindrical shell to be? [Note: If you actually download the article, you will see that end effects from the finite-length cylindrical shields are quite apparent.]

(c) Part (a) assumed the permeability μ is some constant, independent of $|\vec{H}_0|$. In reality, ferromagnetic substances are characterized by a hysteresis curve (see, e.g., Figure 5.12 of Jackson). For very large values of $|\vec{H}_0|$, would you expect a thin cylindrical shell to be an effective magnetic shield? Why or why not?

(d) Now in a continuation of part (a), suppose you add a second long cylindrical shell of the same magnetic material (i.e., same permeability μ), and the same thickness t, but of a larger radius $R' > R \gg t$. That is, the configuration is two co-axial nested cylindrical shields, with an air gap between them (i.e., $R' - R > t$). Find now an expression for the shielding factor, where you again compare the field in the interior region of the smaller-radius cylinder, $|\vec{H}|$, to $|\vec{H}_0|$. You should find a relatively simple expression. Is it better to have two cylindrical shells with radii R and $R' > R$ of the same thickness t and separated by an air gap, or one cylindrical shell of radius R of thickness $2t$?

Problem 4 [10 points]: Ring Rotating in a Magnetic Field

Calculate the exponential decay time, τ, in the angular frequency $\omega(t) = \omega_0 \exp(-t/\tau)$ of a thin ring of mass M and radius R which is suspended from a massless string and is initially rotating with an angular frequency ω_0 at $t = 0$ in a uniform horizontal magnetic field of magnitude B (see schematic below). The ring has a conductivity of σ and a small cross-sectional area of πr^2, where we can assume $r^2 \ll R^2$. Assume that at all times the energy dissipated due to resistive Joule heating during one rotational period of the ring is small compared to the ring's rotational kinetic energy, such that you can work with the time-averaged value of the energy dissipated during the period of the ring's rotation (as opposed to its instantaneous values at all times during each period of rotation).



Problem 5 [20 points]: Magnetized Cylinder Moving at Constant Velocity
A long cylinder of radius a has a magnetization $\vec{M}$ perpendicular to its axis.

(a) First assume the cylinder is at rest. Find the $\vec{B}$ and $\vec{H}$ fields everywhere in space. The cylinder is "long", so you can treat this as a two-dimensional problem.

(b) Let $\hat{z}$ be aligned with the cylinder axis, and let $\hat{x}$ be aligned with the magnetization. Now suppose the cylinder is moving with constant velocity $\vec{v} = v\hat{z}$, with $v \ll c$ such that we can approximate $\vec{B} = \vec{B}'$ (i.e., the magnetic field in the two references frames is the same[1]). Under this approximation, find the resulting charge density and electric field everywhere in the lab frame.

Problem 6 [10 points]: Self Inductance

(a) Show that the self inductance L of any current distribution can be written as:

$$L = \frac{1}{I^2} \int d^3x \; \vec{J}(\vec{x}) \cdot \vec{A}(\vec{x}),$$

where $\vec{A}(\vec{x})$ is the vector potential.

(b) Consider a wire with a finite cross sectional radius of a which extends from $z = -b$ to $z = +b$. The wire carries a current of I which flows in the $+\hat{z}$ direction. Write an expression for the current density $\vec{J}$.

(c) Write an integral expression, in cylindrical coordinates, for the vector potential $\vec{A}(\vec{x})$. You do not need to evaluate this integral. However, make sure to specify the limits on the integrals.

(d) Write another integral expression for the self inductance L of this wire. You can write your integral expression for this part in terms of your integral expression for part (c). Again, make sure to specify the limits on the integrals. [In principle, we could attempt to evaluate these integrals using the methods developed in Section 3.11 of Jackson for the expansion of Green functions in cylindrical coordinates. This would be a difficult exercise. You can look at one such solution in a paper posted at `http://arxiv.org/pdf/1204.1486.pdf` .]

Problem 7 [10 points]: Mutual Inductance
Derive an expression for the mutual inductance of two circular, coaxial loops of radii a and b separated by a distance x, in terms of K and E, the complete elliptic integrals of the first and second kind.

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[1] As will be discussed in PHY 613, the electric and magnetic fields (written in cgs units) transform according to

$$\begin{aligned}
\vec{E}' &= \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{E}), \\
\vec{B}' &= \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{B}),
\end{aligned}$$

where $\vec{\beta} = \vec{v}/c$ and γ is the usual Lorentz factor. Thus, if $v \ll c$, $\gamma \approx 1 + \frac{1}{2}\beta^2$ and if we then neglect $\mathcal{O}(\beta^2)$ terms, we find $\vec{E}' = \vec{E} + \vec{\beta} \times \vec{B}$ and $\vec{B}' = \vec{B} - \vec{\beta} \times \vec{E}$. You should then be able to convince yourself that it is reasonable to approximate $\vec{B} \approx \vec{B}'$ for the parameters of this particular problem if you again neglect terms of $\mathcal{O}(\beta^2)$.