Hydrogen atom wavefunction example:

$$\psi_{\underset{n\,l\,m}{2,1,-1}} = R_{\underset{n\,l}{21}} \cdot Y_{\underset{l\,m}{1,-1}} = \frac{1}{2\sqrt{6a_0^3}}\frac{r}{a_0}e^{-r/2a_0} \cdot \sqrt{\frac{3}{8\pi}}\sin\theta \cdot e^{-i\phi}$$

satisfies:

$$\underbrace{\frac{-\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\psi}_{\circledast} - \frac{\hbar^2}{2\mu r^2}\underbrace{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\psi + \frac{1}{\sin^2\theta}\overbrace{\frac{\partial^2}{\partial\phi^2}}^{\sim L^2_{z\,op}}\psi\right]}_{\sim L^2_{op}}$$

$$+ \frac{kZe^2}{r}\psi = E\psi$$

a) $L_z^2\, e^{-i\phi} = -\hbar^2\frac{\partial^2}{\partial\phi^2}e^{-i\phi} = +\hbar^2 e^{-i\phi}$ so $L_z^2\psi = \hbar^2\psi$

b) $$L^2 \sin\theta e^{-i\phi} = \frac{-\hbar^2}{\sin\theta}\underbrace{\frac{\partial}{\partial\theta}\sin\theta\underbrace{\frac{\partial}{\partial\theta}\sin\theta}_{\cos\theta}}_{\cos^2\theta + -\sin^2\theta = 1-2\sin^2\theta} e^{-i\phi} + \frac{\hbar^2}{\sin^2\theta}\sin\theta e^{-i\phi}$$

$=$

$$= 2\hbar^2\sin\theta\cdot e^{-i\phi}$$

$$L^2\psi_{1,-1} = \hbar^2\cdot\overbrace{1\cdot 2}^{l(l+1)}\psi_{1,-1}$$

c) $$\frac{\partial}{\partial r}\left(re^{-r/2a_0}\right) = \left(\frac{\partial}{\partial r}r\right)e^{-r/2a_0} + r\left(\frac{\partial}{\partial r}e^{-r/2a_0}\right)$$

$$= e^{-r/2a_0} + r\cdot\frac{-1}{2a_0}e^{-r/2a_0}$$

$$\frac{\partial}{\partial r}r^2\left(e^{-r/2a_0} + r\frac{-1}{2a_0}e^{-r/2a_0}\right) = \frac{\partial}{\partial r}\left(r^2 - \frac{r^3}{2a_0}\right)e^{-r/2a_0}$$

$$= \left(2r - \frac{3r^2}{2a_0}\right)e^{-r/2a_0} + \left(r^2 - \frac{r^3}{2a_0}\right)\cdot\frac{-1}{2a_0}e^{-r/2a_0}$$

$$= \left(\frac{r^3}{4a_0^2} - \frac{4r^2}{2a_0} + 2r\right)e^{-r/2a_0} \Rightarrow \circledast$$

$$\frac{-\hbar^2}{2\mu}\frac{1}{r^2}(\circledast) + \left(\frac{\hbar^2\cdot 2}{2\mu r^2} - \frac{kZe^2}{r}\right)re^{-r/2a_0} = E\cdot re^{-r/2a_0}$$

$$\frac{-\hbar^2}{2\mu}\left(\frac{r}{4a_0^2} - \frac{4}{2a_0} + \cancel{\frac{2}{r}}\right) + \frac{\hbar^2}{2\mu}\left(\cancel{\frac{2}{r}}\right) - \frac{kZe^2}{r}\cdot r = E\cdot r$$

comparing powers of r:

r^0: $\frac{\hbar^2}{\mu a_0} = kZe^2$

r^1: $\frac{-\hbar^2}{2\mu\cdot 4a_0^2} = E$

$$\boxed{a_0 = \frac{\hbar^2}{\mu\cdot kZe^2} \qquad E = \frac{\mu(kZe^2)^2}{2\cdot \overset{n}{2^2}\hbar^2}}$$