1. [20 pts] Draw the energy levels and corresponding wavefunctions for the following potentials: a) \( V = \frac{-19.4}{|x| + 1} \) b) 3 square wells of width 1.0 and depth 15.0, using http://www.benfold.com/sse/shoot.html. Write down the numerical value of \( E_n \) for the first six energy eigenvalues. Note the \( n^{th} \) wavefunction should cross the \( x \)-axis \( n - 1 \) times.

2. [20 pts] Write down the Schrödinger equation for a particle in an infinite square well on a cube of dimensions \( a \times b \times c \), i.e. \( V(x, y, z) = 0 \) if \( 0 < x < a \) and \( 0 < y < b \) and \( 0 < z < c \); \( V(x, y, z) = \infty \) elsewhere. What is the value of \( \psi(x, y, z) \) outside of the well? Perform separation of variables to obtain equations for each coordinate \( x, y, z \) inside the well. Solve the equation for \( x \), normalize the solution, and apply the boundary conditions to obtain quantization. Infer the solutions of the other two equations to obtain the total wavefunction \( \psi(x, y, z) \). Calculate the energy and degeneracy of the lowest five energy levels if \( a = b = c = L \) and draw an energy level diagram.

3. [20 pts] Consider the step potential \( V(x) = V_0 \theta(x) \), i.e. \( V(x) = V_0 \) if \( x > 0 \) and \( V(x) = 0 \) if \( x < 0 \). What type of force does this potential describe? Show that \( \psi(x) = e^{\pm ikx} \) are solutions of the Schrödinger equation for this potential in the region 1 (\( x < 0 \)) and region 2 (\( x > 0 \)), and calculate \( k_i \) in regions \( i = 1, 2 \) in terms of the total energy \( E \). To describe the reflection and transmission of a quantum particle, let the total wavefunction be \( \psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \) if \( x < 0 \) and \( \psi(x) = Ce^{ik_2x} \) if \( x > 0 \). Label the incident, transmitted, and reflected wave functions. Apply the boundary conditions at \( x = 0 \) to obtain formulas for the coefficients of reflection \( R \equiv (\frac{B}{A})^2 \) and transmission \( T \equiv \frac{k_2}{k_1} (\frac{C}{A})^2 \) as a function of \( E/V_0 \) (the factor of \( k_2/k_1 \) accounts for the difference in velocity). Show that \( R + T = 1 \), i.e. the total probability for either transmission or reflection is 100%.

4. [20 pts] Substitute the \( \theta \) and \( \phi \) parts of the following \( Y_{lm}(\theta, \phi) \) functions into equations 6.12 and 6.13 and show that they are solutions. Note, this shows they are eigenfunctions of \( \hat{L}^2 \) and \( \hat{L}_z \):

\[
\hat{L}_z Y_{lm} = -i\hbar \frac{\partial}{\partial \phi} Y_{lm} = \hbar m Y_{lm} \quad \text{and} \\
\hat{L}^2 Y_{lm} = \left( \frac{-\hbar^2}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left( -\hbar^2 \frac{\partial^2}{\partial \phi^2} \right) \right) Y_{lm} = \hbar^2 l(l+1) Y_{lm}.
\]

a) \( Y_{00} = 1 \), b) \( Y_{10} = \cos \theta \), c) \( Y_{1\pm 1} = \sin \theta e^{\pm i\phi} \), d) \( Y_{2\pm 1} = \sin \theta \cos \theta e^{\pm 2i\phi} \).

Show that the corresponding orbitals in rectangular coordinates are solutions of \( \nabla^2 Y_{lm} = 0 \) (where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \)):  

a) \( Y_{00} = 1 \) b) \( rY_{10} = z \), c) \( rY_{1\pm 1} = x \pm iy \), d) \( r^2 Y_{2\pm 1} = z(x \pm iy) \).

Note: the real and imaginary components can be separate solutions.