* Realms of Mechanics

- **Classical Mechanics**
  - Particles, waves
  - Mechanics, mechanics

- **Quantum Mechanics**
  - Quantum mechanics

- **Special Relativity**
  - Light

- **Quantum Field Theory**

* Electric Charge (duFay, Franklin)

- **Equal & Opposite**
  - $Q + Q' = 0$

- **Quantized**
  - $Q_e = 1.6 \times 10^{-19} C$

- **Locally Conserved**
  - Continuity

* Electric Force (Coulomb, Cavendish)

* Electric Field (Faraday)

- **Action at a Distance**
  - Field "mediates" or carries force

- **Divergence**
  - Field line density

* Equipotential Surfaces / Flow

- **No Work Done**
  - To field lines

- **Circulation or EMF**
  - In a closed loop

* Unification of Forces

- **Neutonian Gravity**
  - $G = \frac{G M_1 M_2}{R^2}$

- **Maxwellian E&M**
  - $E = \nabla \psi$

- **Weak Decays**
  - $Z^0 \rightarrow \mu^+ \mu^-$

- **Nuclear Force**
  - $F = \frac{G_N}{r^2}$

* Electric Potential

<table>
<thead>
<tr>
<th>Force Field</th>
<th>Energy Potential</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F = qE$</td>
<td>$U = q \phi$</td>
</tr>
<tr>
<td>$F = mg$</td>
<td>$U = mgh$</td>
</tr>
</tbody>
</table>

* Electric Flux

- **Flux = # of Field Lines**
- **Density of Lines = Field Strength**
- **Divergence**
- **Electric Flux Density**

- **Note:** $\frac{\Phi}{\text{area}} = \Omega = \text{independent of distance}$

- **Electric Flux**
  - Flows from $(+)$ to $(-)$
  - All flux lines begin at $+$ and end at $-$

- **Potential if Flow is Independent of Path**
  - Circulation or EMF in a closed loop

* Survey of Electromagnetism

- **General Relativity**
  - Curved spacetime

- **Classical Mechanics**
  - Quantum mechanics

- **Quantum Mechanics**
  - Particle/wave duality

- **Quantum Field Theory**
  - Second quant.
  - Gauge fields
  - $U^+, U^-$

- **Electric Potential**
  - No work done to field lines
  - Equipotentials = surfaces of const energy
  - Work is done along field line
  - Flow = # of potential surfaces crossed
Magnetic field
- No magnetic charge (monopole)
- Field lines must form loops
- Permanent magnetic dipoles first discovered
  - Torque: \( \mathbf{r} = \mathbf{r} \times \mathbf{B} \)
  - Energy: \( U = -\mathbf{r} \cdot \mathbf{B} \)
  - Force: \( F = \nabla \times (\mathbf{r} \cdot \mathbf{B}) \)

- Electric current shown to generate fields (Oersted, Ampere)
- Magnetic dipoles are current loops
- Biot-Savart law - analog of Coulomb law

\[
\mathbf{r} = \int \mathbf{r} \times \frac{\mu_0}{4\pi} \int \mathbf{r} \times \frac{\mu_0}{r^2} \]

- \( B = \) flux density
- \( H = \) field intensity
  \( \mathbf{B} = \mu \mathbf{H} = \frac{\mathbf{B}}{\mathbf{A}} \)

Faraday law
- Opposite of Oersted's discovery:
  - Changing magnetic flux induces potential (EMF)
- Electric generators, transformers

\[
\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

Maxwell equations
- Added displacement current - \( D \) lines have +/- charge at each end
- Changing displacement current equivalent to moving charge
- Derived conservation of charge and restored symmetry in equations
- Predicted electromagnetic radiation at the speed of light

\[
C = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}
\]

Maxwell equations
\[
\nabla \cdot \mathbf{B} = 0
\]
\[
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{J}
\]

Constitutive equations
\[
\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}
\]

Lorentz force
\[
\mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q (\mathbf{E} + \mathbf{v} \times \mathbf{B})
\]

Continuity
\[
\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0
\]

Potentials
\[
\mathbf{E} = -\nabla \Phi \quad \mathbf{B} = \nabla \times \mathbf{A}
\]

Gauge transformation
\[
V \rightarrow V - \Omega \quad \mathbf{A} \rightarrow \mathbf{A} + \mathbf{v} \lambda
\]

Wave equation
\[
-\nabla^2 (V \mathbf{A}) = (\rho \mathbf{E}, \mu \mathbf{J})
\]
Section 1.1 - Vector Algebra

* Linear spaces
  ~ linear combination: \((a\vec{a} + b\vec{b})\) is the basic operation
  ~ basis: \((\hat{a}, \hat{b}, \hat{c})\) # basis elements = dimension
  independence: not collapsed into lower dimension
  closure: vectors span the entire space

~ components: \(\vec{X} = a_\alpha \hat{a}_\alpha + b_\beta \hat{b}_\beta + c_\gamma \hat{c}_\gamma\)
  in matrix form:
  \[
  \begin{pmatrix}
  X \\
  y \\
  z
  \end{pmatrix}
  =
  \begin{pmatrix}
  a_x & a_y & a_z \\
  b_x & b_y & b_z \\
  c_x & c_y & c_z
  \end{pmatrix}
  \begin{pmatrix}
  \alpha \\
  \beta \\
  \gamma
  \end{pmatrix}
  \]
  where \(\vec{X} = \hat{X} a_\alpha + \hat{Y} b_\beta + \hat{Z} c_\gamma\)

~ Einstein notation: implicit summation over repeated indices
~ direct sum: \(C = A \oplus B\) add one vector from each independent
  space to get vector in the product space (not simply union)

~ projection: the vector \(\vec{X} = \hat{a} + \hat{b}\) has a unique decomposition
  (coordinates \((a, b)\) in \(A, B\)) - relation to basis/components?
  ~ all other structure is added on as multilinear (tensor) extensions

* Metric (inner, dot) product - distance and angle

  \(C = \vec{a} \cdot \vec{b} = ab \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\)

~ properties:
  1) scalar valued - what is outer product?
  2) bilinear form
  3) symmetric

~ orthonormality and completeness - two fundamental identities
  help to calculate components, implicitly in above formulas

\[
\begin{array}{c}
\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \\
\sum_{i=1}^3 \hat{e}_i \cdot \hat{e}_i = I
\end{array}
\]

\(x = b_x x^i = b_x \hat{e}_i\)

\(a_i = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3\)

~ orthogonal projection: a vector \(\hat{n}\) divides the space \(X\) into \(X_\parallel \oplus X_\perp\)
  geometric view: dot product \(\hat{n} \cdot \hat{x}\) is length of \(\hat{x}\) along \(\hat{n}\)
  Projection operator: \(P_\parallel = \hat{n} \hat{n} \cdot \) acts on \(x\): \(P_\parallel \hat{x} = \hat{n} \hat{n} \cdot \hat{x}\)

~ generalized metric: for basis vectors which are not orthonormal,
  collect all nxn dot products into a symmetric matrix (metric tensor)

\[
g_{ij} = \begin{pmatrix}
\hat{e}_i \cdot \hat{e}_j & \hat{e}_i \cdot \hat{b}_j & \hat{e}_i \cdot \hat{c}_j \\
\hat{b}_i \cdot \hat{e}_j & \hat{b}_i \cdot \hat{b}_j & \hat{b}_i \cdot \hat{c}_j \\
\hat{c}_i \cdot \hat{e}_j & \hat{c}_i \cdot \hat{b}_j & \hat{c}_i \cdot \hat{c}_j
\end{pmatrix}
\]

\(x \cdot \hat{y} = x^i \hat{e}_i \cdot \hat{b}_j x^j = x^i g_{ij} x^j \)

\(= x^T \begin{pmatrix}
g_{ij} & \hat{b}_i \\
\hat{c}_i & \hat{b}_j
\end{pmatrix} x\)

in the case of a non-orthonormal basis, it is more difficult to find components of a vector, but it can be accomplished using the reciprocal basis (see HW1)
Exterior Products - higher-dimensional objects

* cross product (area)

\[ \mathbf{c} = \mathbf{a} \times \mathbf{b} = \hat{n} \, \mathbf{a} \times \mathbf{b} \sin \theta = \hat{n} \mathbf{a} \times \mathbf{b} = \hat{n} \mathbf{a} \mathbf{b} = \hat{n} \mathbf{a} \mathbf{b} \]

where \( \hat{n} \perp \mathbf{a} \) and \( \hat{n} \perp \mathbf{b} \) (RH-rule)

\[
\begin{vmatrix}
\hat{a}_x & \hat{a}_y & \hat{a}_z \\
\hat{b}_x & \hat{b}_y & \hat{b}_z \\
\hat{c}_x & \hat{c}_y & \hat{c}_z
\end{vmatrix}
\]

~ properties:
1) vector-valued
2) bilinear
3) antisymmetric \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \)

~ components:
\[
\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \, \mathbf{e}_k
\]

where \( \varepsilon_{ij3} = \varepsilon_{j3i} = \varepsilon_{3ij} = 1 \)
\( \varepsilon_{i3j} = \varepsilon_{3ij} = -1 \)

Levi-Civita tensor - completely antisymmetric:

\[
\mathbf{x} \times \mathbf{y} = \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \varepsilon^{ijk} \mathbf{x}_i \mathbf{y}_j \mathbf{e}_k
\]

~ orthogonal projection: \( \hat{n} \times \mathbf{x} \) projects \( \mathbf{x} \) to \( \hat{n} \) and rotates by 90°

\[ \mathbf{P}_1 = -\hat{n} \times \hat{n} \mathbf{x} \]

where is the metric in x?

* triple product (volume of parallelepiped) - base times height

~ completely antisymmetric - definition of determinant

~ why is the scalar product symmetric / vector product antisymmetric?

~ acts more like a 'trivector' (volume element)

~ again, where is the metric? (not needed!)

* exterior algebra (Grassman, Hamilton, Clifford)

~ extended vector space with basis elements from objects of each dimension

~ pseudo-vectors, scalar separated from normal vectors, scalar

~ magnitude, length, area, volume

~ scalar, vectors, bivectors, trivector

\[ \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l \]

~ what about higher-dimensional spaces (like space-time)?

~ can’t form a vector ‘cross-product’ like in 3-d, but still have exterior product

~ all other products can be broken down into these 8 elements

most important example: BAC-CAB rule (HW: relation to projectors)

\[ \mathbf{A} \mathbf{B} \mathbf{C} \mathbf{D} = \mathbf{A} \mathbf{B} \mathbf{C} \mathbf{D} - \mathbf{D} \mathbf{B} \mathbf{C} \mathbf{A} \]

\[
\varepsilon_{ijk} \mathbf{A}^i \mathbf{B}^j \mathbf{C}^k = \left( \delta^m_{i} \delta^l_{i} - \delta^m_{i} \delta^l_{i} \right) \mathbf{A}^i \mathbf{B}^m \mathbf{C}^l = \mathbf{B}^i \left( \mathbf{A}^l \mathbf{C}^j \right) - \mathbf{C}^l \left( \mathbf{A}^i \mathbf{B}^j \right)
\]
**Section 1.1.5 - Linear Operators**

* Linear Transformation
  ~ function which preserves linear combinations
  ~ determined by action on basis vectors (egg-crate)
  ~ rows of matrix are the image of basis vectors
  ~ determinant = expansion volume (triple product)
  ~ multilinear (2 sets of bases) - a tensor

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* Change of coordinates
  ~ two ways of thinking about transformations both yield the same transformed components
  ~ active: basis fixed, physically rotate vector
  ~ passive: vector fixed, physically rotate basis

* Transformation matrix (active) - basis vs. components

\[
\begin{align*}
(\hat{a} \hat{b} \hat{c}) &= \left(\begin{array}{c} \hat{x} \\ \hat{y} \end{array}\right) \left(\begin{array}{c} a_x \\ a_y \\ b_x \\ b_y \\ c_x \\ c_y \end{array}\right) \\
\hat{x}' &= \left(\begin{array}{c} a \ b \\ c \ d \end{array}\right) \left(\begin{array}{c} \hat{x} \\ \hat{y} \end{array}\right) \\
\hat{e}' &= \hat{e} \mathbf{R} \quad \hat{x}' = \hat{e}' \mathbf{R} = \hat{e} \mathbf{R} \hat{x} = \hat{x} = \mathbf{R} \hat{x}' \end{align*}
\]

* Orthogonal transformations
  ~ R is orthogonal if it 'preserves the metric' (has the same form before and after)

\[
\begin{align*}
\hat{e}'^T \hat{e}' &= \left(\begin{array}{c} \hat{x}' \\ \hat{y}' \end{array}\right)^T \left(\begin{array}{c} \hat{x}' \\ \hat{y}' \end{array}\right) = (\hat{x}' \hat{x}' + \hat{y}' \hat{y}') = g' \\
\hat{e}^T \hat{e} &= \left(\begin{array}{c} \hat{x} \\ \hat{y} \end{array}\right)^T \left(\begin{array}{c} \hat{x} \\ \hat{y} \end{array}\right) = (\hat{x} \hat{x} + \hat{y} \hat{y}) = g \\
\hat{e}' &= \hat{e} \mathbf{R} \quad \hat{x}' = \hat{e}' \mathbf{R} = \hat{e} \mathbf{R} = \mathbf{R} g \mathbf{R} = g' \\
\mathbf{R}^T g \mathbf{R} &= g' \quad g = g'
\end{align*}
\]

* Symmetric / antisymmetric vs. Symmetric / orthogonal decomposition
  ~ recall complex numbers \( U = \rho + i \phi \quad \rho^2 = \rho \quad (i \phi)^* = -i \phi \)
  ~ \( e^U = e^{\rho + i \phi} = r e^{i \phi} \quad |e^{i \phi}|^2 = e^{-i \phi} e^{i \phi} = e^{0} = 1 \)
  ~ similar behaviour of symmetric / antisymmetric matrices

\[
\begin{align*}
M &= \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = (a (b+c) \frac{1}{a} + (b-c) \frac{1}{c}) + (c \frac{1}{b} + 0) = T + A \\
e^M &= 1 + M + \frac{1}{2!} M^2 + \frac{1}{3!} M^3 + ... = e^{T+A} \\
S &= e^T = e^{WV^T} = e^W e^{V^{-1}} \\
R &= e^A \quad \mathbf{R}^T = e^{T+A} = e^A \\
\det(e^A) &= \det(e^{a_1 e^{a_2} e^{...} e^{a_{n-1}} e^{a_n}}) = e^{\prod a_i} \\
\det(e^A) &= e^{\text{tr} A} = e^0 = 1
\end{align*}
\]
**Eigenparaphernalia**

* illustration of symmetric matrix $S$ with eigenvectors $v$, eigenvalues $\lambda$

\[
S v = \lambda v \\
(2 1)(v_1) = \lambda (v_1) \\
(2 1)(v_2) = \lambda (v_2) \\
(2 1)(0) = (3) = \lambda (0) \\
(2 1)(1) = (3) = \lambda (1)
\]

~ what about an antisymmetric/orthogonal matrix?

* similarity transform - change of basis (to diagonalize $A$)

\[
S (v_1 v_2 \ldots) = (v_1 v_2 \ldots) (\lambda_1 \lambda_2 \ldots) \quad S V = V W V^{-1} = V W V^T
\]

* a symmetric matrix has real eigenvalues

\[
S v = \lambda v \\
v^T S v = \lambda v^T v \\
v^T S = v^T \lambda^* \\
v^T S V = \lambda^* v^T V \\
\lambda = \lambda^*
\]

~ what about a antisymmetric/orthogonal matrix?

* eigenvectors of a symmetric matrix with distinct eigenvalues are orthogonal

\[
v^T S = (S v)^T = (S V)^T = (\lambda v)^T = v^T \lambda \\
\lambda_1 v_1 \perp v_2 = (V_1^T S) v_2 = V_1^T S V_2 = V_1 \perp v_2 \lambda \\
v_1 \cdot v_2 (\lambda_1 - \lambda_2) = 0 \quad \text{if } \lambda_1 \neq \lambda_2 \text{ then } v_1 \cdot v_2 = 0.
\]

* singular value decomposition (SVD)

~ transformation from one orthogonal basis to another

\[
M = RS = RVWV^T = UWV^T
\]

~ extremely useful in numerical routines

- $M$ arbitrary matrix
- $R$ orthogonal
- $S$ symmetric
- $W$ diagonal matrix
- $V$ orthogonal (domain)
- $U$ orthogonal (range)
Section 1.2 - Differential Calculus

* differential operator

~ ex. \( u = x^2 \) \( \quad \frac{du}{dx} = 2x \)

~ \( \frac{d}{dx} (\sin x^2) = 2x \cos x^2 \cdot 2x dx \)

~ \( df \) and \( dx \) connected - refer to the same two endpoints

~ made finite by taking ratios (derivative or chain rule)

~ or infinite sum = integral (Fundamental Theorems of Calculus)

\[
\frac{df}{dx} = \frac{df}{du} \frac{du}{dx} \quad \quad \int_a^b \frac{df}{dx} \, dx = \int_a^b df = f \big|_a^b
\]

* scalar and vector fields - functions of position \( \mathbf{r} \)

~ "field of corn" has a corn stalk at each point in the field

~ scalar fields represented by level curves (2d) or surfaces (3d)

~ vector fields represented by arrows, field lines, or equipotentials

* partial derivative & chain rule

~ signifies one varying variable and other fixed variables

~ notation determined by denominator; numerator along for the ride

~ total variation split into sum of variations in each direction

\[
\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} \quad \cdots = \frac{dx}{\partial x} + \frac{dy}{\partial y} + \frac{dz}{\partial z}
\]

* vector differential - gradient

~ differential operator \( \nabla \)

\[
\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}
\]

~ differential line element: \( d\mathbf{l} \) and \( d\mathbf{l} \) transforms between \( \mathbf{x},\mathbf{y},\mathbf{z} \leftrightarrow dx,dy,dz \) and \( d \leftrightarrow \nabla \)

~ example:

\[
\frac{\partial^2}{\partial x^2} y = 2x dy + x^2 dy = (2xy, x^2) \cdot (dx, dy)
\]

~ example: let \( z = f(x,y) \) be the graph of a surface. What direction does \( \nabla f \) point?

now let \( g = z - f(x,y) \) so that \( g = 0 \) on the surface of the graph

then \( \nabla g = (-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}) \) is normal to the surface

* illustration of curl

* illustration of divergence
Higher Dimensional Derivatives

* curl - circular flow of a vector field
\[ \nabla \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{z}(V_{y,z} - V_{x,y}) - \hat{y}(V_{z,x} - V_{x,z}) + \hat{x}(V_{y,x} - V_{y,y}) \]

* divergence - radial flow of a vector field
\[ \nabla \cdot \vec{V} = \left( \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right) \left( \begin{array}{c} V_x \\ V_y \\ V_z \end{array} \right) = V_{x,x} + V_{y,y} + V_{z,z} \]

* product rules
  ~ how many are there?
  ~ examples of proofs
\[ \begin{align*}
\vec{A} \times (\vec{B} \times \vec{C}) &= \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \\
\vec{A} \times (\nabla \times \vec{B}) &= \nabla(\vec{A} \cdot \vec{B}) - \vec{B}(\vec{A} \cdot \nabla) \\
\vec{A} \times (\nabla \times \vec{B}) &= \vec{B}(\vec{A} \cdot \nabla) - \vec{A}(\vec{B} \cdot \nabla) \\
\vec{A} \times (\nabla \times \vec{B}) &= \nabla \times (\vec{A} \times \vec{B}) - \vec{B}(\vec{A} \cdot \nabla) - \vec{A}(\vec{B} \cdot \nabla) \\
\end{align*} \]

* second derivatives - there is really only ONE! (the Laplacian)
\[ \nabla^2 \equiv \nabla \nabla \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

1) \[ \nabla \cdot \nabla^2 = \nabla^2 \nabla \]
  ~ eg: \[ \nabla^2 \vec{T} = 0 \] no net curvature - stretched elastic band
  ~ defined component-wise on \( v_x, v_y, v_z \) (only cartesian coords)

3), 5) \[ \nabla^2 \vec{V} = \nabla \times \nabla \vec{V} \]
  ~ longitudinal / transverse projections
\[ \nabla \cdot \nabla^2 \vec{V} = \nabla \times \nabla \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{z}(\partial^2_{yy} - \partial^2_{zz}) - \hat{y}(\partial^2_{zz} - \partial^2_{xx}) + \hat{x}(\partial^2_{xx} - \partial^2_{yy}) \]

* unified approach to all higher-order derivatives with differential operator
1) \[ d^2 = 0 \]
2) \[ dx^2 = 0 \]
3) \[ dx \, dy = -dy \, dx \]
  ~ Gradient
\[ d \vec{f} = d_x \vec{f} + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \nabla \vec{f} \cdot d\vec{l} \\
\]
  ~ Divergence
\[ d(\vec{A} \cdot d\vec{l}) = d(A_x dx + A_y dy + A_z dz) = (A_{x,x} dx^2 + A_{x,y} dy dx + A_{x,z} dz dx) + (A_{y,x} dy dx + A_{y,y} dy^2 + A_{y,z} dy dz) + (A_{z,x} dz dx + A_{z,y} dz dy + A_{z,z} dz^2) \]
\[ d(\vec{A} \cdot d\vec{l}) = (\nabla \times \vec{A}) \cdot d\vec{a} \]
\[ d\vec{a} = (dx \, dy, dx \, dz, dy \, dz) = \hat{z} dx \times dy \, dz = dx \, dy \, dz \]

\[ \nabla \vec{f} = \frac{df}{dx} = \frac{df}{dy} = \frac{df}{dz} \]

\[ \nabla \vec{A} = \frac{d(\vec{A} \cdot d\vec{l})}{d\vec{a}} = \frac{d(\vec{f} \cdot d\vec{A})}{d\vec{a}} \]

\[ \nabla \cdot \vec{B} = \frac{d(\vec{B} \cdot d\vec{l})}{d\vec{a}} = \frac{d(\vec{f} \cdot \vec{B})}{d\vec{a}} \]
**Section 1.4 - Affine Spaces**

* Affine Space - linear space of points

**POINTS vs VECTORS**

- operations
  \[
  Q - P = \vec{V} \\
  P + \vec{V} = Q
  \]
  \[
  \vec{W} = \alpha \vec{U} + \beta \vec{V}
  \]

- points are invariant under translation of the origin
- can treat points as vectors from the origin to the point
- cumbersome picture: many meaningless arrows from meaningless origin
- position field point \( \vec{p} = (x, y, z) \)
- vector: source pt \( \vec{p}' = (x', y', z') \)
- the only operation on points is the weighted average
- weight \( w = 0 \) for vectors and \( w = 1 \) for points
- transformation: affine vs linear
- decomposition: coordinates vs components
  - they appear the same for cartesian systems!
  - coordinates are scalar fields \( q^i(\vec{r}) \)

* Rectangular, Cylindrical and Spherical coordinate transformations

- math: \( 2d \rightarrow N-d \) physics: 3d + azimuthal symmetry
- singularities on z-axis \( (\) and origin

rect.  cyl.  sph.

\[
\begin{align*}
\chi &= S \cdot \cos \phi = r \cdot \sin \theta \cdot \cos \phi \\
y &= S \cdot \sin \phi = r \cdot \sin \theta \cdot \sin \phi \\
z &= z &= r \cdot \cos \theta
\end{align*}
\]

\[
\begin{align*}
dl_{\text{rec}} &= \hat{x} dx + \hat{y} dy + \hat{z} dz \\
dl_{\text{cyl}} &= \hat{r} dr + \hat{\phi} \cdot s d\phi + \hat{z} dz \\
dl_{\text{sph}} &= \hat{r} dr + \hat{\theta} \cdot s d\theta + \hat{\phi} \cdot s^2 d\phi \\
d\vec{a}_{\text{rec}} &= \hat{x} \, dy \, dz + \hat{y} \, dz \, dx + \hat{z} \, dx \, dy \\
d\vec{a}_{\text{cyl}} &= \hat{s} \, d\phi \, dz + \hat{\phi} \, d\phi \, ds + \hat{z} \, ds \, d\phi \\
d\vec{a}_{\text{sph}} &= \hat{r} \, d\theta \, s^2 d\phi + \hat{\theta} \, s^2 d\phi \, dr + \hat{\phi} \, s^2 d\phi \, d\theta \\
dt_{\text{rec}} &= dx \, dy \, dz \\
dt_{\text{cyl}} &= ds \cdot d\phi \cdot dz \\
dt_{\text{sph}} &= dr \cdot r d\theta \cdot r \sin \theta d\phi \\
\]

- level surface
Curvilinear Coordinates

* coordinate surfaces and lines
  ~ each coordinate is a scalar field \( \phi(\mathbf{r}) \)
  ~ coordinate surfaces: constant \( \phi^i \)
  ~ coordinate lines: constant \( \phi^j, \phi^k \)

* coordinate basis vectors
  \[ \mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial \phi^i} \]
  \[ \mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial \phi^i} \]
  \[ \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \]
  \[ \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \]

* differential elements

** formulas for vector derivatives in curvilinear coordinates

\[
\begin{align*}
\nabla \mathbf{\phi} &= \frac{\partial \mathbf{\phi}}{\partial \phi^i} \cdot \partial_i = \nabla \mathbf{\phi} \\
\n\nabla \cdot \mathbf{\phi} &= \frac{\partial \mathbf{\phi}}{\partial \phi^i} \cdot \partial_i = \nabla \cdot \mathbf{\phi} \\
\n\nabla \times \mathbf{\phi} &= \frac{\partial \mathbf{\phi}}{\partial \phi^i} \cdot \partial_i = \nabla \times \mathbf{\phi} \\
\n\nabla^2 \mathbf{\phi} &= \frac{1}{h_i h_j} \left[ \nabla \cdot \left( \nabla \mathbf{\phi} \right) \right] - \partial_i \partial_i \mathbf{\phi} \\
\n\nabla^2 \mathbf{\phi} &= \frac{1}{h_i h_j} \left[ \nabla \cdot \left( \nabla \mathbf{\phi} \right) \right] - \partial_i \partial_i \mathbf{\phi} \\
\end{align*}
\]
* different types of integration in vector calculus

1-dim: \( \omega^1 = \lambda \, ds \), \( \sigma \, ds \), \( \delta \cdot ds \), \( \delta \cdot d\theta \)

2-dim: \( \omega^2 = \sigma \, da \), \( \sigma \, da \), \( \delta a \cdot da \), \( \delta a \cdot \delta a \)

3-dim: \( \omega^3 = \rho \, dt \), \( \delta \cdot dt \)

~ "differential forms" are the things after the
all have a 'd' somewhere inside

~ often \( d\delta, d\sigma, d\lambda \) are buried inside of another 'd'

~ current element \( d\sigma = q^{(i)} \cdot \nabla \cdot \delta \), \( \delta \cdot d\theta \)

~ charge element \( d\delta = \nabla \cdot \delta \), \( \delta \cdot d\theta \)

~ two types of regions:

  over the region \( R \): \( \int_R \omega \) (open region)

  over the boundary \( \partial R \) of \( R \): \( \int_{\partial R} \omega \) (closed region)

* recipe for ALL types of integration

a) Parametrize the region

~ parametric vs relations equations of a region

~ boundaries translate to endpoints on integrals

b) Pull back the parameters

~ \( x,y,z \) become functions of \( s,t,u \)

~ differentials: \( dx,dy,dz \) become \( ds,dt,du \)

~ reduce using the chain rule

\[
\int_A \delta \cdot d\lambda = \int_A \alpha \left( x,y,z \right) dx + A_y \left( x,y,z \right) dy + A_z \left( x,y,z \right) dz \\
= \int_{\partial A} \alpha \left( x(t),y(t),z(t) \right) \frac{dx}{dt} dt + A_y \left( x(t),y(t),z(t) \right) \frac{dy}{dt} dt + A_z \left( x(t),y(t),z(t) \right) \frac{dz}{dt} dt
\]

c) Integrate 1-d integrals using calculus of one variable

* example: line & surface integrals on a paraboloid (Stoke's theorem)

\[
\begin{align*}
\int_S x^2 + y^2 & \quad S: \quad z = \frac{1}{2} x^2 + y^2 = \frac{1}{2} (x^2 + y^2) \\
\int_{S \leq 2} & \quad \delta S = 1 = \frac{1}{2} x^2 + y^2 \\
\int_{x=2s_2} \delta x & \quad = 2 ds_2 c_y + 2 s_2 s_3 d\phi \\
\int_{y=3s_3} \delta y & \quad = ds_3 s_4 + s c_y d\phi \\
\int_{z=s^2} \delta z & \quad = 2 s_4 ds_3 + s c_y d\phi \\
\int_{\delta \sigma} & \quad = \int_{\delta \sigma} \frac{2 s_4}{2 s_4} ds_3 d\phi \\
\int_{\delta \delta} & \quad = -2 s_4 s_3 + \frac{2 s_4 d\phi}{2 s_4} = -2 s_4 s_3 \\
\int_{\delta \lambda} & \quad = \int_{\partial s} y \cdot ds = -2 \int_{s_3} s_3 d\phi = -2 \pi
\end{align*}
\]

* alternate method: substitute for \( dx, dy, d\phi \) (antisymmetric)

\[
\begin{align*}
\int_S y \cdot dx & \quad = \int_S s c_y \cdot (2s_4 ds_3 - 2 s_3 s_4 d\phi) \\
\int_S y \cdot dy & \quad = \int_S s c_y \cdot (2s_4 ds_3 - 2 s_3 s_4 d\phi) \\
\int_S y \cdot d\phi & \quad = \int_S s c_y \cdot (2s_4 ds_3 - 2 s_3 s_4 d\phi)
\end{align*}
\]
**Flux, Flow, and Substance**

* **Differential forms**

<table>
<thead>
<tr>
<th>Name</th>
<th>Geometrical picture</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>scalar:</strong></td>
<td>( \Phi(\mathbf{x}) = \Phi(\mathbf{r}) )</td>
</tr>
<tr>
<td><strong>vector:</strong></td>
<td>( \mathbf{A} \cdot d\mathbf{A} = A_x , dx + A_y , dy + A_z , dz )</td>
</tr>
<tr>
<td><strong>pseudovector:</strong></td>
<td>( \mathbf{B} \cdot d\mathbf{A} = B_x , dy , dz + B_y , dz , dx + B_z , dx , dy )</td>
</tr>
<tr>
<td><strong>pseudoscalar:</strong></td>
<td>( \Phi = \rho , dt = \rho , dx , dy , dz )</td>
</tr>
</tbody>
</table>

* **Derivative 'd'**

<table>
<thead>
<tr>
<th>Name</th>
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</tr>
</thead>
<tbody>
<tr>
<td><strong>scalar:</strong></td>
<td>( \nabla \Phi = \triangledown \Phi \cdot d\mathbf{A} )</td>
</tr>
<tr>
<td><strong>vector:</strong></td>
<td>( \nabla \times \mathbf{A} = \nabla \times \mathbf{A} \cdot d\mathbf{A} )</td>
</tr>
<tr>
<td><strong>pseudovector:</strong></td>
<td>( \nabla \times \mathbf{B} = \mathbf{B} \cdot d\mathbf{A} )</td>
</tr>
<tr>
<td><strong>pseudoscalar:</strong></td>
<td>( \nabla \times \Phi = \Phi , dt )</td>
</tr>
</tbody>
</table>

* **Definite integral**

<table>
<thead>
<tr>
<th>Name</th>
<th>Geometrical picture</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>scalar:</strong></td>
<td>( \mathbb{E} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{A} )</td>
</tr>
<tr>
<td><strong>vector:</strong></td>
<td>( \mathbb{E} = \oint_{\partial S} \mathbf{B} \cdot d\mathbf{A} )</td>
</tr>
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<td><strong>pseudovector:</strong></td>
<td>( \mathbb{E} = \int_V \Phi )</td>
</tr>
<tr>
<td><strong>pseudoscalar:</strong></td>
<td>( \mathbb{E} = \int_V \Phi )</td>
</tr>
</tbody>
</table>

* **Stokes' theorem**

  # of flux tubes puncturing disk (S) bounded by closed path
  
  **EQUALS** # of surfaces pierced by closed path (\( \partial S \))
  
  ~ each surface ends at its SOURCE flux tube

* **Divergence theorem**

  # of substance boxes found in volume (R) bounded by closed surface
  
  **EQUALS** # of flux tubes piercing the closed surface (\( \partial R \))
  
  ~ each flux tube ends at its SOURCE substance box
Section 1.3.2-5 - Region | Form = Integral

* Regions

~ definition of boundary operator \( \partial \)

'closed' region (cycle): \( \partial S = 0 \)

~ a room (walls, window, ceiling, floor)

is CLOSED if all doors, windows closed

is OPEN if the door or window is open;

~ what is the boundary?

~ think of a surface that has loops

that do NOT wrap around disks!

* Forms - see last notes

~ combinations of scalar/vector fields and differentials so they can be integrated

~ pictorial representation enables 'integration by eye'

<table>
<thead>
<tr>
<th>RANK</th>
<th>NOTATION</th>
<th>REGION</th>
<th>VISUAL REP.</th>
<th>DERIVATIVE</th>
</tr>
</thead>
<tbody>
<tr>
<td>scalar</td>
<td>( w^{(0)} = f )</td>
<td>( \partial ) Q point</td>
<td>level surfaces</td>
<td>( d \omega^{(0)} = \nabla f \cdot d\ell )</td>
</tr>
<tr>
<td>vector</td>
<td>( w^{(1)} = \vec{A} \cdot d\vec{l} )</td>
<td>( \partial ) P path</td>
<td>flow sheets</td>
<td>( d \omega^{(1)} = \nabla \times \vec{A} \cdot d\vec{a} )</td>
</tr>
<tr>
<td>p-vector</td>
<td>( w^{(2)} = \vec{B} \cdot d\vec{S} )</td>
<td>( \partial ) S surface</td>
<td>flux tubes</td>
<td>( d \omega^{(2)} = \nabla \cdot \vec{B} \cdot d\vec{S} )</td>
</tr>
<tr>
<td>p-scalar</td>
<td>( w^{(3)} = r )</td>
<td>( \partial ) V volume</td>
<td>subst boxes</td>
<td>( d \omega^{(3)} = 0 )</td>
</tr>
</tbody>
</table>

~ properties of differential operator \( \partial \)

transforms form into higher-dimensional form, sitting on the boundary

~ Poincare lemma \( \partial \partial \omega = 0 \)

\( \nabla \times \nabla \omega = 0 \)

\( \nabla \cdot \nabla \times \vec{A} = 0 \)

~ converse - existence of potentials \( \nabla \vec{A} \)

\( d \omega = 0 \iff \omega = d\alpha \)

\( \nabla \times \vec{E} = 0 \iff \vec{E} = \nabla \Phi \)

\( \nabla \cdot \vec{B} = 0 \iff \vec{B} = \nabla \times \vec{A} \)

for space without any n-dim 'holes' in it

* Integrals - the overlap of a region on a form = integral of form over region

~ regions and forms are dual - they combine to form a scalar

~ generalized Stoke's theorem:

'\( \partial \)' and '\( \partial \)' are adjoint operators - they have the same effect in the integral

\[ \int_{\partial R} \omega = \int_R d\omega \]

\[ \int_{\partial R} \omega = \int_R d\omega = \int_R dd\omega = 0 \]
**Generalized Stokes Theorem**

* Fundamental Theorem of Vector Calculus: 0d → 1d
\[ \int_{a}^{b} \nabla \phi \cdot d\mathbf{x} = \int_{a}^{b} df = f(b) - f(a) \]

* Stokes' Theorem: 1d → 2d
\[ \nabla \mathbf{A} \cdot d\mathbf{a} = \frac{\partial A_y}{\partial x} \, dx \, dy + \frac{\partial A_x}{\partial y} \, dy \, dx + \ldots \]
\[ = A_y(x') \, dy + A_y(x) \, (-dy) + A_x(y') \, (-dx) + A_x(y) \, dx + \ldots \]
\[ = \sum \mathbf{A} \cdot d\mathbf{l} \text{ around boundary} \]
\[ + \text{ other faces} \]

* Gauss' Theorem: 2d → 3d (divergence theorem)
\[ \nabla \cdot \mathbf{B} \, d\tau = \frac{\partial B_x}{\partial x} \, dx \, dy \, dz + \frac{\partial B_y}{\partial y} \, dy \, dx \, dz + \frac{\partial B_z}{\partial z} \, dz \, dx \, dy \]
\[ = B_x(x') \, dy \, dz + B_x(x) \, (-dy) \, dz + 4 \text{ other faces} \]
\[ = \sum \mathbf{B} \cdot d\mathbf{a} \text{ around boundary} \]

* Note: all interior \( f(x) \), flow, and flux cancel at opposite edges
* Proof of converse Poincare lemma: integrate form out to boundary
* Proof of gen. Stokes theorem: integrate derivative out to the boundary

\[ \int_{\mathbf{R}} d\mathbf{\omega} = \oint_{\mathbf{R}} \phi \cdot d\mathbf{a} = \oint_{\mathbf{P}} \frac{\partial \phi}{\partial \mathbf{P}} \]
\[ \oint_{\mathbf{S}} \mathbf{A} \cdot d\mathbf{a} = \oint_{\mathbf{S}} \mathbf{B} \cdot d\mathbf{a} \]
\[ \oint_{\mathbf{U}} \mathbf{B} \cdot d\mathbf{c} = \oint_{\mathbf{U}} \mathbf{B} \cdot d\mathbf{a} \]

* Example - integration by parts
\[ \nabla \cdot \left( \frac{\hat{e}}{r^2} \phi \right) = \left( \nabla \cdot \frac{\hat{e}}{r^2} \right) \phi + \frac{\hat{e}}{r^2} \cdot \nabla \phi \]
\[ \int_{\mathbf{r}} \phi \cdot d\tau = \int_{\mathbf{r}} \nabla \cdot \left( \frac{\hat{e}}{r^2} \phi \right) \cdot d\tau - \int_{\mathbf{v}} \left( \nabla \cdot \frac{\hat{e}}{r^2} \right) \phi \cdot d\tau \]
\[ \int_{\mathbf{v}} \frac{1}{r^2} \frac{\partial \phi}{\partial r} \, r^2 \, dr \, d\mathbf{\omega} = \int_{\mathbf{v}} d\mathbf{\omega} \cdot \frac{\hat{e}}{r^2} \phi - \int_{\mathbf{v}} 4\pi \, \delta^3(\mathbf{r}) \phi \cdot d\tau \]
\[ \int_{\mathbf{v}} d\mathbf{\omega} \int_{r_{\mathbf{v}}}^{\mathbf{R}} df = \int_{r_{\mathbf{v}}}^{\mathbf{R}} d\mathbf{\omega} \hat{e} \cdot \frac{\hat{e}}{r^2} f - 4\pi \, f(0) \]
\[ \int_{\mathbf{v}} d\mathbf{\omega} \int_{r_{\mathbf{v}}}^{\mathbf{R}} f(R_{\mathbf{v}}, \theta, \phi) = 4\pi \, f(0) \]
\[ 4\pi \left[ \langle f \rangle_{\mathbf{R}} - f(0) \right] = 4\pi \left[ \langle f \rangle_{\mathbf{R}} - f(0) \right] \]
Section 1.5 - Dirac Delta Distribution

* Newton's law: \( \text{yank} = \text{mass} \times \text{jerk} \)

\[ \frac{d}{dt} \frac{dy}{dt} = \frac{d^2y}{dt^2} \]

http://wikipedia.org/wiki/position_%28vector%29

* important integrals related to \( \delta(x) \)

\[ \int_{-\infty}^{\infty} \delta(x) f(x) \, dx = f(0) \]

\[ \int_{-\infty}^{\infty} \delta'(x) f(x) \, dx = f'(0) \]

* \( \delta(x-x') \) is the an "undistribution" - it integrates to a lower dimension

\[ \int \delta(x) \, dx = \int \delta(t) \, dt = q \]

\[ \int \delta'(x) \, dx = \int \delta'(t) \, dt = q \]

\[ \int \delta''(x) \, dx = \int \delta''(t) \, dt = q \]

or \[ \int q \delta(x-x') \, dx = q \quad \text{or} \quad \int q \delta'(x-x') \, dx = q \]

* \( \delta(x-x') \) gives rise to boundary conditions - integrate the diff. eq. across the boundary

\[ \nabla \cdot D = \rho = \gamma(s,t) \delta(n) \]

\[ \nabla \rightarrow \nabla \cdot \Delta \quad \rho \rightarrow \sigma \quad f \rightarrow \mathbf{R} \]

* \( \delta(x-x') \) is the "kernel" of the identity transformation

\[ f = \mathbf{I} \cdot f \]  

\[ f(x) = \int_{-\infty}^{\infty} f'(x') \delta(x-x') \, dx' \]

* \( \delta(x-x') \) is the continuous version of the "Kroneker delta" \( \delta_{ij} \)

\[ a_1 = \sum_{j=1}^{3} \delta_{ij} a_j \quad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \]
Linear Function Spaces

* functions as vectors (Hilbert space)
  ~ functions under pointwise addition have the same linearity property as vectors

VECTORS
  ~ addition \( \mathbf{W} = \mathbf{V} + \mathbf{U} \) \( \mathbf{W}_i = \mathbf{V}_i + \mathbf{U}_i \)
  \[ \mathbf{v} = \sum_{i=1}^{\infty} V_i \hat{e}_i \]
  or \( \mathbf{v} = \sum_{i=1}^{\infty} \hat{e}_i f_i \phi_i(x) \)

FUNCTIONS
  \[ h = f + g \quad h(x) = f(x) + g(x) \]
  \[ f(x) = \int_{-\infty}^{\infty} \mathbf{f}(x') \delta(x - x') dx' \]
  or \( f(x) = \int_{-\infty}^{\infty} \mathbf{f}_i \phi_i(x) dx' \)

~ graph

~ addition

~ expansion

~ inner product

~ orthonormality

~ closure

~ linear operator

~ orthogonal rotation

~ eigen-expansion
  (stretches) (principle axes) (Sturm-Liouville problems)

~ gradient,
  functional derivative

* Sturm-Liouville equation - eigenvalues of function operators (2nd derivative)

\[ L[y] = -\frac{d}{dx} \left[ p(x) \frac{d}{dx} y \right] + q(x) y = \lambda \omega(x) y \]

BC: \( y(a), y(b) \)

~ there exists a series of eigenfunctions \( y_i(x) \) with eigenvalues \( \lambda_n \)
~ eigenfunctions belonging to distinct eigenvalues are orthogonal \( \langle y_i, y_j \rangle = \delta_{ij} \)
Green Functions $G(x, y)$

Green's functions are used to "invert" a differential operator
- they solve a differential equation by turning it into an integral equation

You already saw them last year! (in Phy 232)
- the electric potential of a point charge

\[ \nabla \cdot \frac{\hat{F}}{r^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\hat{F}}{r^2} \right) = 0 \]

a) \( \frac{1}{r^2} \to \infty \) at \( r=0 \) "singularity"

b) \( \int_\Omega \frac{\hat{F}}{r^2} \, dr = \hat{F} \, d\Omega \cdot \frac{\hat{F}}{r^2} = \frac{\partial}{\partial r} \frac{1}{r^2} \int_\Omega \hat{F} \, dr = 4\pi \)

- independent of volume if \( \hat{F} \) inside

Thus \( \nabla \cdot \frac{\hat{F}}{r^2} = 4\pi \delta^3(\hat{F}) \)

\[ \nabla \cdot \frac{\hat{F}}{r^2} = \frac{\partial}{\partial r} \left( r^2 \frac{\hat{F}}{r^2} \right) = -\frac{\hat{F}}{r^2} \]

\[ \nabla \nabla \cdot \frac{\hat{F}}{r^2} \to \frac{\partial}{\partial r} \frac{1}{r^2} \int_\Omega \hat{F} \, dr = 4\pi \delta^3(\hat{F}) \]

\[ -\nabla^2 V = \frac{\partial}{\partial eps} \]

\[ V \to E \to \Phi/\epsilon_0 \]

\[ \frac{1}{r} \to \frac{F}{r^2} \to 4\pi \delta^3(\hat{F}) \]

\[ \text{(Poisson equation)} \]

Green's functions are the simplest solutions of the Poisson equation

\[ G(r, \hat{F}) = \frac{-1}{4\pi^2} = \nabla^2 \delta^3(\hat{F}) \]

- is a special function which can be used to solve Poisson equation symbolically
- using the "identity" nature of \( \delta^3(\hat{F}-\hat{F}) = \delta^3(\hat{F}) \)

- intuitively, it is just the "potential of a point source"

\[ \nabla^2 G(r) = \nabla \cdot \nabla \frac{-1}{4\pi^2} = \nabla \cdot \frac{\hat{F}}{4\pi^2} = \delta^3(\hat{F}) \quad \hat{F} \equiv \hat{F} - \hat{F}' \]

let \( V = \int_V G(r) \rho(\hat{F}) \, d\tau' \quad (\text{Solution to Poisson's eq.}) \)

\[ \nabla^2 V = \int_V \frac{-\rho(\hat{F})}{\epsilon_0} \nabla^2 G(\hat{F}, \hat{F}') \, d\tau' = \int_V \frac{-\rho(\hat{F})}{\epsilon_0} \delta^3(\hat{F}, \hat{F}') \, d\tau' = -\frac{\rho(\hat{F})}{\epsilon_0} \]

This generalizes to one of the most powerful methods of solving problems in E&M
- in QED, Green's functions represent a photon 'propagator'
- the photon mediates the force between two charges
- it 'carries' the potential from charge to the other

\[ U = \int \rho V \, d\tau = \int \int \rho G \rho \, d\tau \, d\tau' \]

\[ \rho \xrightarrow{G} G(\hat{F}) \xrightarrow{\rho} \]

\[ e \xrightarrow{\rho} r \xrightarrow{e} \]

\[ \text{Diagram of photon propagation} \]
Section 1.6 - Helmholtz Theorem

* orthogonal projections $P_\parallel$ and $P_\perp$: a vector $\hat{n}$ divides the space $\mathbb{R}^3$ into $\mathbb{R}^3_\parallel \oplus \mathbb{R}^3_\perp$ geometric view: dot product $\hat{n} \cdot \vec{x}$ is length of $\vec{x}$ along $\hat{n}$

Projection operator: $P_\parallel = \hat{n} \hat{n} \cdot$, acts on $\vec{x}$: $P_\parallel \vec{x} = \hat{n} \hat{n} \cdot \vec{x}$

~ orthogonal projection: $\hat{n} \times$ projects $\perp$ to $\hat{n}$ and rotates by 90°

$$\hat{n}_\perp = -\hat{n} \times (\hat{n} \times \vec{x}) = P_\perp \vec{x} \quad P_\perp = -\hat{n} \times \hat{n} \times$$

$$P_\parallel + P_\perp = \hat{n} \hat{n} \cdot -\hat{n} \times \hat{n} \times = I$$

* longitudinal/transverse separation of Laplacian (Hodge decomposition)

$$\nabla^2 \vec{F} = \vec{P} \quad \nabla \times \nabla \cdot \vec{F} - \nabla \times \nabla \times \vec{F}$$

(prolongational/longitudinal components of $\nabla$)

~ formal, $\vec{F} = -\nabla(-\nabla^2 \vec{F}) + \nabla \times (-\nabla^2 \nabla \times \vec{F})$

$\rho, \vec{J}$ are SOURCES $\psi, \vec{A}$ are POTENTIAL

~ what does $\nabla^2 \vec{F}$ mean? Note that $-\nabla^2 \frac{1}{4\pi r} = \delta^3(\vec{x})$

~ thus $\nabla^2 \delta^3(\vec{x}) = -\frac{1}{4\pi r} \equiv G(\vec{x})$ (see next page)

$G = -\frac{1}{4\pi r}$ is Green fn

~ use the $\delta$-identity $\rho(\vec{r}) = \int d\tau \delta^3(\vec{x}) \rho(\vec{r})$

$V(\vec{r}) = -\nabla^2 \rho(\vec{r}) = \int d\tau (\nabla^2 \delta^3(\vec{x})) \rho(\vec{r}) = \int d\tau \frac{\rho(\vec{r})}{4\pi \tau} = \frac{1}{4\pi \epsilon_0} \int d\vec{r} \rho(\vec{r})$

$\vec{A}(\vec{r}) = -\nabla^2 \vec{J}(\vec{r}) = \int d\tau (\nabla^2 \delta^3(\vec{x})) \vec{J}(\vec{r}) = \int d\tau \frac{\vec{J}(\vec{r})}{4\pi \tau} = \frac{\mu_0}{4\pi} \int d\tau \vec{J}(\vec{r})$

~ thus any field can be decomposed into L/T parts

SCALAR POTENTIAL $V$

* Theorem: the following are equivalent definitions of an "irrotational" field:
  a) $\nabla \times \vec{F} = 0$ curl-less
  b) $\vec{F} = -\nabla V$ where $V = \int \frac{\vec{F} \cdot d\vec{A}}{4\pi r}$$
  c) $V(\vec{r}) = \int_{-\infty}^{\vec{r}} \frac{\vec{F} \cdot d\vec{A}}{4\pi r}$ is independent of path
  d) $\int \vec{F} \cdot d\vec{A} = 0$ for any closed path

* Gauge invariance:
  if $\vec{F} = -\nabla V_1$ and also $\vec{F} = -\nabla V_2$
  then $\nabla (V_2 - V_1) = 0$ and $V_2 |_{\tau} = V_1$ is constant ("ground potential")

VECTOR POTENTIAL $\vec{A}$

* Theorem: the following are equivalent definitions of a "solenoidal" field:
  a) $\nabla \cdot \vec{F} = 0$ divergence-less
  b) $\vec{F} = \nabla \times \vec{A}$ where $\vec{A} = \int \frac{\delta^3(\vec{x}) \vec{F}}{4\pi \tau}$
  c) $\vec{F} = \int \frac{\delta^3(\vec{x}) \vec{F}}{4\pi \tau}$ with $\delta^3(\vec{x})$ fixed
  d) $\int \vec{F} \cdot d\vec{A} = 0$ for any closed surface

* Gauge invariance:
  if $\vec{F} = \nabla \times \vec{A}_1$ and also $\vec{F} = \nabla \times \vec{A}_2$
  then $\nabla \times (\vec{A}_2 - \vec{A}_1) = 0$ and $\vec{A}_2 |_{\tau} = \nabla \times \lambda(\vec{r})$ ("gauge transformation")
Section 2.1 - Coulomb's Law

- Electric charge (duFay, Franklin)
  - \( +, - \) equal & opposite (QCD: \( r + g + b = 0 \))
  - \( e = 1.6 \times 10^{-19} \) C, quantized \( (\phi < 2 \times 10^{-21} e) \)
  - locally conserved

- only for static charge distributions (test charge may move but not sources)

a) Coulomb's law
\[ F = \frac{1}{4\pi \varepsilon_0} \frac{qQ}{r^2} \]

b) Superposition
\[ F = \sum F_i \]

- Born-Infeld:
  - vacuum polarization violates superposition at the level of \( \alpha = \frac{1}{137.2} \)

- Electric field
  - we want a vector field, but \( F \) only at test charge
  - action at a distance: the field 'carries' the force from source pt. to field pt.

- Example (Griffiths Ex. 2.1)

\[ \oint_{q_0} d\mathbf{q} = \sigma d\mathbf{a} = \sigma d\mathbf{a} \cdot \mathbf{r}^2 \]

\[ F = \frac{1}{4\pi \varepsilon_0} \left( \frac{q_0 q}{r^2} + \frac{q_0 q}{r^2} + \ldots \right) Q = Q \mathbf{E} \]

\[ E = \frac{1}{4\pi \varepsilon_0} \sum \frac{q_i q}{r_i^2} = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\mathbf{r}) d\mathbf{r}}{r^2} = \frac{1}{4\pi \varepsilon_0} \int \frac{d\mathbf{r}}{d^2} \]

\[ d\mathbf{q} \rightarrow q_i = q(\mathbf{r}_i) \text{ or } \lambda(\mathbf{r}_i) d\mathbf{r} \text{ or } \sigma(\mathbf{r}_i) d\mathbf{a} \text{ or } \rho(\mathbf{r}_i) d\mathbf{r} \]

\[ \int_{x=0}^{L} \frac{d\mathbf{q}'}{r^2} = \frac{1}{4\pi \varepsilon_0} \int_{0}^{L} \frac{2\lambda d\mathbf{x}' \cdot \mathbf{z} \cdot \hat{z}}{(z^2 + x^2)^{3/2}} + O \]

\[ = \frac{2}{4\pi \varepsilon_0} \int \frac{\sec^2 \theta d\theta}{\sec^2 \theta} \]

\[ = \frac{2}{4\pi \varepsilon_0} \int \sec^2 \theta d\theta \]

\[ = \frac{2}{4\pi \varepsilon_0} \sin \theta \bigg|_{\theta = 0}^{\theta = \pi} \]

\[ = \frac{2}{4\pi \varepsilon_0} \frac{L}{\sqrt{z^2 + x^2}} \]

\[ = \frac{2}{4\pi \varepsilon_0} \frac{L}{\sqrt{z^2 + 1}} \]

\[ \approx \frac{2}{4\pi \varepsilon_0} \frac{L}{z^2} \text{ as } z \rightarrow \infty \]

\[ \approx \frac{2}{4\pi \varepsilon_0} \frac{L}{z} \text{ as } z \rightarrow \infty \]
Section 2.2 - Divergence and Curl of E

- 5 formulations of electrostatics
  - Gauss' law
  - Flux (Equipotential surfaces)
  - Flow (Field lines)
  - Potential

- Integral field eq's
  - \( \Phi_E = Q/\varepsilon_0 \)
  - \( \mathcal{E}_E = 0 \) (closed regions)
  - Conservation of flux

- Differential field eq's
  - \( \nabla \cdot \vec{E} = \rho/\varepsilon_0 \)
  - \( \nabla \times \vec{E} = 0 \)

- Divergence theorem: relationship between differential and integral forms of Gauss' law
  \[ \Phi_E = \int_{\partial \Omega} \vec{E} \cdot d\vec{a} = \frac{q}{4\pi\varepsilon_0} \cdot \mathcal{E}_E \mathcal{V} = \frac{q}{\varepsilon_0} \rightarrow \int_{\Omega} \frac{q}{\varepsilon_0} \, d\mathcal{V} \]
  \[ \int_{\Omega} \nabla \cdot \vec{E} \, d\mathcal{V} = \int_{\partial \Omega} \mathcal{E}_E \, d\mathcal{S} \]

- Since this is true for any volume, we can remove the integral from each side
  \[ \nabla \cdot \vec{E} = \frac{\mathcal{E}_E}{\varepsilon_0} \]
Section 2.3 - Electric Potential

* two personalities of a vector field: Flux $= \Phi_E = \int_S \mathbf{E} \cdot d\mathbf{a}$ (streamlines) through an area
  Dr. Jekyl and Mr. Hyde
  Flow $= \mathbf{E} = \int_S \mathbf{E} \cdot d\mathbf{a}$ (equipotentials) downstream

* direct calculation of flow for a point charge
  $\mathbf{E}_E = \int_V \mathbf{E} \cdot d\mathbf{a} = \int_V \frac{\mathbf{E}_Q}{4\pi \varepsilon_0} \cdot d\mathbf{a}$
  note: this is a perfect differential (gradient)
  $\frac{\mathbf{E}_Q}{4\pi \varepsilon_0} \cdot d\mathbf{a} = \mathbf{E}_Q \cdot d\mathbf{a}$
  $\mathbf{E} = \mathbf{E}_Q \cdot d\mathbf{a}$
  $\mathbf{E}_{\text{open path}} = \mathbf{E}_{\text{closed path}}$  (constant of integration)
  $\mathbf{E} = -\nabla V$
  $\mathbf{E}_E = \int_S \mathbf{E} \cdot d\mathbf{a}$

~ open path: note that this integral is independent of path
  thus $\mathbf{V}(\mathbf{r}) = -\mathbf{E}_E = \int_S \mathbf{E} \cdot d\mathbf{a}$ is well-defined
  by FITV:
  $\Delta V = \int_S \nabla \cdot \mathbf{E} \cdot d\mathbf{a}$
  so $\mathbf{E} = -\nabla V$
  $\mathbf{E}_E = -\nabla V$

~ Poincaré lemma: if $\mathbf{E} = -\nabla V$ then $\nabla \times \mathbf{E} = -\nabla \times \nabla V = 0$

~ converse: if $\nabla \times \mathbf{E} = 0$ then $\mathbf{E} = -\nabla V$

* Poisson equation $\nabla \cdot \mathbf{E} = -\nabla \cdot \varepsilon_0 \nabla V = \rho$
  or $\nabla^2 V = \rho / \varepsilon_0$
  ~ next chapter devoted to solving this equation - often easiest for real-life problems
  ~ a scalar differential equation with boundary conditions on $E$ or $V$
  ~ inverse (solution) involves:
    a) the solution for a point charge (Green's function)
    $\mathbf{V}(\mathbf{r}) = \int_S \frac{\rho(\mathbf{r}')}{4\pi \varepsilon_0} \mathbf{r} \cdot d\mathbf{a}'$
    $\nabla^2 G(x, \mathbf{r}) = \delta^3(x - \mathbf{r})$
    $G(x, \mathbf{r}) = \nabla^2 \delta^3(x - \mathbf{r})$
    $\rho(\mathbf{r}) = \int_V \rho(\mathbf{r}') d\mathbf{a}'$
    $\mathbf{V}(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{4\pi \varepsilon_0} \mathbf{r} \cdot d\mathbf{a}'$
  b) an arbitrary charge distribution is a sum of point charges (delta functions)

~ this is an essential component of the Helmholtz theorem
  $\mathbf{E} = -\nabla (-\nabla^2 \mathbf{E})$
  $\nabla \times (-\nabla^2 \mathbf{E}) = -\nabla \times (-\nabla^2 \mathbf{E})$
  thus $\mathbf{E} = -\nabla V$  $\Rightarrow$  $\nabla \times \mathbf{E} = 0$

* derivative chain
  $\mathbf{V} \rightarrow \mathbf{E} \rightarrow \mathbf{E}_E \rightarrow \mathbf{E}_S$
  ~ inverting Gauss' law is more tortuous path!
  $\rho \rightarrow \mathbf{V} \rightarrow \mathbf{E} \\ \mathbf{E} = -\nabla V = \int_S \frac{\rho_0(x)}{4\pi \varepsilon_0} \mathbf{n} \cdot d\mathbf{a}$
  $\mathbf{E} = -\nabla V = \int_S \frac{\rho_0(x)}{4\pi \varepsilon_0} \mathbf{n} \cdot d\mathbf{a}$
**Field Lines and Equipotentials**

* for along an equipotential surface:
  
  * field lines are normal to equipotential surfaces

* dipole "two poles" - the word "pole" has two different meanings: (but both are relevant)
  
  a) opposite (+ vs -, N vs S, bi-polar)
  b) singularity (1/r has a pole at r=0)

* effective monopole (dominated by -2q far away)

* quadrupole (compare HW3 #2)
* show that $\nabla \cdot \vec{E} = \rho / \varepsilon_0$ from Coulomb's law

\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left( \frac{\partial}{\partial x-x'}, \frac{\partial}{\partial y-y'}, \frac{\partial}{\partial z-z'} \right) = \nabla \text{ (if } x' \text{ fixed)} \]

\[ \nabla \cdot \int \frac{\rho(r') \, dt'}{4\pi \varepsilon_0 \cdot r^2} = \nabla \cdot \int \frac{\rho(r') \, dt'}{4\pi \varepsilon_0 \cdot r^2} = \frac{1}{4\pi \varepsilon_0} \int \rho(r') \, dt' \nabla \cdot \frac{\vec{E}}{r^2} \]

\[ = \frac{1}{4\pi \varepsilon_0} \int \rho(r') \, dt' 4\pi \delta^3(r') = \rho(r') / \varepsilon_0 \]

* derive Coulomb's law from the differential field equations

\[ \nabla \cdot \vec{E} = \rho / \varepsilon_0 \quad \nabla \times \vec{E} = 0 \quad \nabla^2 = \nabla \cdot \nabla \quad \nabla \times \nabla \times \vec{E} = 0 \]

\[ \vec{E} = -\nabla (-\nabla \cdot \vec{E}) + \nabla \times (\nabla \times \vec{E}) = -\nabla \int \frac{\rho(r') \, dt'}{4\pi \varepsilon_0} = -\nabla \int \frac{\rho(r') \, dt'}{4\pi \varepsilon_0} \frac{\vec{E}}{r^2} \]

\[ = \int \frac{\rho(r') \, dt'}{4\pi \varepsilon_0} \vec{E} \quad \frac{\vec{E}}{r^2} = \int \frac{\rho(r') \, dt'}{4\pi \varepsilon_0} \frac{\vec{E}}{r^2} = \int \frac{\rho(r') \, dt'}{4\pi \varepsilon_0} \frac{\vec{E}}{r^2} \]

* show that the differential and integral field equations are equivalent

\[ \Phi_E = \frac{Q}{\varepsilon_0} \iff \nabla \cdot \vec{E} = \rho / \varepsilon_0 \quad \Phi_E = \int_{s} \vec{E} \cdot d\vec{a} = \int_{\partial s} \vec{E} \cdot d\vec{a} \quad \frac{Q}{\varepsilon_0} = \int_{\partial s} \frac{\rho}{\varepsilon_0} \, dt \]

~ apply the divergence theorem

~ since Gauss' law holds for any volume, it is only true if the integrands are equal

* Griffiths 2.6 find potential of spherical charge distribution

\[ \int \vec{E} \cdot d\vec{a} = \int \frac{\rho}{\varepsilon_0} \, dt \quad 4\pi r^2 \quad \frac{E(r)}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \quad \text{if } r > r' \]

if $r > r'$

\[ V(r) = \int_{r'}^{r} E(r') \, dr' = \int_{r'}^{r} \frac{\rho(r')}{4\pi \varepsilon_0 r'^2} \, dr' = \frac{Q}{4\pi \varepsilon_0 r} \]

if $r < r'$

\[ V(r) = V(r') + \int_{r}^{r'} E(r') \, dr' = V(r') + \int_{r}^{r'} \frac{\rho(r')}{4\pi \varepsilon_0 r'^2} \, dr' \]

* Griffiths 2.7 integrate potential due to spherical charge distribution

\[ 4\pi \varepsilon_0 \int_{u=1}^{u} \frac{d\sigma}{\varepsilon_0} \quad \frac{d\sigma}{
abla} = \frac{r'^2}{4\pi} \sin^2 \phi' \, d\phi' \]

\[ = \frac{1}{2} \int_{u=1}^{u} \frac{d\sigma}{\varepsilon_0} \quad \frac{d\sigma}{
abla} = \frac{r'^2}{4\pi} \sin^2 \phi' \, d\phi' = \frac{r'^2}{4\pi} \sin^2 \phi' \, d\phi' \]

\[ = \frac{r'^2}{2 \pi} \left( 1 - (1-u) + (1+u) \right) \]

\[ = \frac{r'^2}{2 \pi} \left( 1 - r'^2 + r'^4 \right) \quad r > r' \]

\[ = \frac{r'^2}{2 \pi} \left( 1 - r'^2 + r'^4 \right) \quad r < r' \]

\[ V(r) = \frac{Q}{4\pi \varepsilon_0} \frac{1}{r} \quad \text{if } r > r' \]

\[ V(r) = \frac{Q}{4\pi \varepsilon_0} \frac{1}{r} \quad \text{if } r < r' \]
* Griffiths 2.8 find the energy due to a spherical charge distribution

\[ W = \frac{1}{2} \int \sigma \cdot V = \frac{1}{2} \int q \cdot V = \frac{1}{2} \frac{q^2}{4\pi \varepsilon_0 r^1} \]

\[ W = \frac{\varepsilon_0}{2} \int E^2 dV = \frac{\varepsilon_0}{2} \int \int r dr d\theta \left( \frac{q}{(4\pi \varepsilon_0 r^1)} \right)^2 \]

\[ = \frac{q^2}{2 \cdot 4\pi \varepsilon_0} \int \int \frac{dr}{r^2} = \frac{q^2}{2 \cdot 4\pi \varepsilon_0 r^1} \]

* Quiz: calculate field at origin from a hemispherical charge distribution

\[ E = \int \frac{dq \hat{r}}{4\pi \varepsilon_0 \ r^2} = \int \int \frac{q}{4\pi \varepsilon_0} d\Omega \ \frac{(-x\hat{x} - y\hat{y} - z\hat{z})}{4\pi \varepsilon_0 R^3} \]

\[ \frac{dq}{d\Omega} = \frac{q}{4\pi \varepsilon_0} d\Omega = \sigma \ d\alpha \]

\[ = \frac{-q \hat{z}}{2\pi \cdot 4\pi \varepsilon_0 R^3} \int_0^{\pi/2} R \cos \theta (-d\cos \theta) \int_0^{2\pi} d\phi = \frac{-q \hat{z}}{8\pi \varepsilon_0 R^2} \]

\[ -R \cos \theta \int_0^{\pi/2} R \cos \theta = \frac{-R}{2} \int_0^{2\pi} \]

\[ \frac{R}{2 \pi} \]
Section 2.4 - Electrostatic Energy

* analogy with gravity

\[
\vec{F} = q \vec{E} \\
W = q \vec{E} \cdot d\vec{l} = m \vec{a} \cdot d\vec{x}
\]

* energy of a distribution of charge \( q_1, q_2, ... \)

\[
W = \frac{1}{4 \pi \varepsilon_0} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{q_i q_j}{r_{ij}} = \frac{1}{4 \pi \varepsilon_0} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{q_i q_j}{r_{ij}}
\]

* continuous version

\[
\sum_{i=1}^{n} q_i \rightarrow \int dq \\
W = \frac{1}{2 \varepsilon_0} \int \rho \nabla \cdot \vec{E} \, d\tau \\
W = \frac{1}{2} \int \rho \frac{d\vec{V}}{dt} \\
W = \frac{1}{2} \int \rho \frac{d\vec{V}}{dt}
\]

* energy density stored in the electric field - integration by parts

\[
\nabla \cdot (\vec{V} \vec{E}) = \nabla \vec{V} \cdot \vec{E} + \vec{V} \nabla \cdot \vec{E} = -\vec{E} \cdot \vec{E} + \varepsilon_0 \rho / \varepsilon_0
\]

\[
0 = \int_{-\infty}^{\infty} d\tau \cdot (\nabla \cdot \vec{E}) = \int_{-\infty}^{\infty} \nabla \cdot (\vec{V} \vec{E}) = \int_{-\infty}^{\infty} \vec{E} \cdot \vec{E} + \varepsilon_0 \rho / \varepsilon_0 \, d\tau
\]

\[
\frac{dW}{dt} = \frac{\varepsilon_0 E^2}{2}
\]

* energy of a point charge in a potential

\[
W = \int_{\alpha}^{\beta} \vec{F} \cdot d\vec{l} = -Q \int_{\alpha}^{\beta} \vec{E} \cdot d\vec{l} = Q \Delta V
\]

\[W(\vec{r}) = Q V(\vec{r}) \quad V(\infty) = 0\]

* work does work follow the principle of superposition

\[
\vec{F} = \vec{F}_1 + \vec{F}_2 = q (\vec{E}_1 + \vec{E}_2) = -q \nabla (V_1 + V_2 + ...)
\]

* energy is quadratic in the fields, not linear

\[
W_{tot} = \frac{\varepsilon_0}{2} \int E^2 \, d\tau = \frac{\varepsilon_0}{2} \int E_1^2 + E_2^2 + 2 \vec{E}_1 \cdot \vec{E}_2 \, d\tau
\]

\[
W_{tot} = W_1 + W_2 + \varepsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau
\]

* the cross term is the interaction energy between two charge distributions

(is the energy stored in the field, or in the force between the charges?)

(is the field real, or just a calculational device?)

(is a tree falls in the forest ...)

* work does work follow the principle of superposition

~ we know that electric force, electric field, and electric potential do

~ energy is quadratic in the fields, not linear

~ the cross term is the 'interaction energy' between two charge distributions

(the work required to bring two systems of charge together)
**Section 2.5 - Conductors**

* conductor
  - has abundant “free charge”, which can move anywhere in the conductor

* types of conductors
  i) metal: conduction band electrons, ~ 1/ atom
  ii) electrolyte: positive & negative ions

* electrical properties of conductors
  i) electric field = 0 inside conductor
     therefore \( V = \) constant inside conductor
  ii) electric charge distributes itself
      all on the boundary of the conductor
  iii) electric field is perpendicular to the surface just outside the conductor

* induced charges
  - free charge will shift around charge on a conductor
  - induces opposite charge on near side of conductor
  - to cancel out field lines inside the conductor
  - Faraday cage: external field lines are shielded
    inside a hollow conductor
  - field lines from charge inside a hollow conductor are
    “communicated” outside the conductor by induction
    (as if the charge were distributed on a solid conductor)
  - compare: displacement currents, sec. 7.3

* electrostatic pressure
  - on the surface: \( \frac{P}{A} = f = \sigma (\vec{E}_{\text{patch}} + \vec{E}_{\text{other}}) = \frac{1}{2} \sigma (\vec{E}_{\text{inside}} + \vec{E}_{\text{outside}}) \)
  - for a conductor: \( \vec{E}_{\text{inside}} = 0 \quad \vec{E}_{\text{outside}} = \frac{\sigma}{\varepsilon_0} \)
  - \( P = f = \frac{\sigma^2}{2\varepsilon_0} = \frac{\sigma E^2}{2} \)
  - note: electrostatic pressure corresponds to energy density
    both are part of the stress-energy tensor
    \( P = \omega \)
Capacitance

* capacitance
~ a capacitor is a pair of conductors held at different potentials, stores charge
~ electric flow from one conductor to the other equals the potential difference
~ electric flux from one conductor to the other is proportional to the charge

\[ C = \frac{Q}{\Delta V} = \frac{\varepsilon \Phi}{E} \quad \Delta V = \int d\Phi \cdot \vec{E} = \varepsilon E \]  
(closed surface)

[diagram of a capacitor with labels for capacitance, potential difference, and electric field]

* work formulation
\[ W = \frac{1}{2} QV = \frac{1}{2} CV^2 = \int \frac{\varepsilon}{2} E^2 \, d\tau \]
\[ C = \frac{2W}{V^2} = \frac{\varepsilon}{V^2} \int E^2 \, d\tau = \frac{\varepsilon}{2} \text{flux} \cdot \text{flow} \]

* capacitance matrix
~ in a system of conductors, each is at a constant potential
~ the potential of each conductor is proportional to the individual charge on each of the conductors
~ proportionality expressed as a matrix

coefficients of potential \( P_{ij} \) or capacitance matrix \( C_{ij} \)

\[
\begin{align*}
V_i &= P_{ij} Q_j \\
Q_i &= C_{ij} V_j
\end{align*}
\]

[diagram showing electric potential and flux]
Section 3.1 - Laplace's Equation

* overview: we learned the math (Ch 1) and the physics (Ch 2) of electrostatics basically all of the concepts of Phy232, but in a new sophisticated language
~ Ch 3: Boundary Value Problems (BVP) with Laplace's equation (NEW!)
  a) method of images  b) separation of variables  c) multipole expansion
~ Ch 4: Dielectric Materials: free and bound charge (more in-depth than 232)

Equations of electrodynamics:
\[ F = q \left( E + \nabla \times B \right) \] Lorentz force
\[ \nabla \cdot j + q = 0 \] Continuity
\[ \nabla \cdot B = \rho \] Maxwell electric, \( \nabla \times E = 0 \) magnetic fields
\[ \nabla \cdot B = 0 \ \nabla \times H - \mu_0 J = 0 \] Constitution
\[ E = -\nabla V - \sigma A \ \ A = \nabla \times A \] Potentials
\[ V \rightarrow V + \alpha \lambda \ \ \lambda \rightarrow \lambda + \nabla \lambda \] Gauge transform

* Classical field equations - many equations, same solution:
Laplace/Poisson: \( \nabla^2 V = 0 \) \( \epsilon \nabla^2 V = \rho \) \( \sim \) potentials \((V, A)\), dielectric \( \epsilon \), permeability \( \mu \)
Maxwell wave: \( \frac{1}{c_0^2} \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \rho \) \( \sim \) speed of light \( c \), charge/current density \((\rho, J)\)
Heat equation: \( \frac{\partial V}{\partial t} = k \nabla^2 V \) \( \sim \) temp \( T \), cond. \( k \), heat \( \dot{Q} = -k \nabla W \) heat cap. \( c \)
Diffusion eq: \( \frac{\partial V}{\partial t} = D \nabla^2 V \) \( \sim \) concentration \( u \), diffusion \( D \), flow \( D \nabla V \)
Drumhead wave: \( \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} - \nabla u = f \) \( \sim \) displacement \( u \), speed of sound \( c \), force \( f \)
Schrödinger: \( \frac{1}{2m} \nabla^2 \psi + i \hbar \frac{\partial \psi}{\partial t} = 0 \) \( \sim \) prob amp \( \psi \), mass \( m \), potential \( V \), Planck \( \hbar \)

* 1-dimensional Laplace equation \( \nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0 \)
\[ \frac{dV}{dx} = \int_0^a \delta(x) \ dx = a \]
\( \sim \) charge singularity between two regions:

\[ V(x) = \frac{1}{2} V(x_0) + \frac{V(x_1)}{2} \]
\( \sim \) mean field:
\( \sim \) no local maxima or minima (stretches tight)

* 2-dimensional Laplace equation \( \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \)
\( \sim \) no straightforward solution (method of solution depends on the boundary conditions)
\( \sim \) Partial Differential Equation (elliptic second order)
\( \sim \) chicken & egg: can't solve \( \frac{\partial^2 V}{\partial y^2} \) until you know \( \frac{\partial V}{\partial y} \)
\( \sim \) solution of a rubber sheet
\( \sim \) no local extrema -- mean field:
\[ V(\Gamma) = \frac{1}{2\pi R} \oint_{\Gamma} V dl \]

* 3-dimensional Laplace equation
\( \sim \) generalization of 2-d case
\( \sim \) same mean field theorem:
\[ V(\Omega) = \frac{1}{4\pi R^2} \int_{\Omega} V d\alpha \]
Boundary Conditions

* 2nd order PDE's classified in analogy with conic sections: replacing $\frac{\partial}{\partial x}$ with $x$, etc
  a) Elliptic - "spacelike" boundary everywhere (one condition on each boundary point)
      eg. Laplace's eq, Poisson's eq.
  b) Hyperbolic - "timelike" (2 initial conditions) and "spacelike" parts of the boundary
      eg. Wave equation
  c) Parabolic - 1st order in time (1 initial condition)
      eg. Diffusion equation, Heat equation

* Uniqueness of a BVP (boundary value problem) with Poisson's equation:
  if $V_1$ and $V_2$ are both solutions of $\nabla^2 V = -\varphi_0$ then let $U = V_1 - V_2$, $\nabla^2 U = 0$
  integration by parts: $\nabla \cdot (U \nabla U) = U \nabla \cdot \nabla U + \nabla U \cdot \nabla U = U \nabla^2 U + (\nabla U)^2$
  in region of interest: $\int_\partial \hat{n} \cdot (U \nabla U) = \int_\partial \hat{n} \cdot U \nabla U + \int_\partial (\nabla U)^2$
  note that: $\nabla^2 U = 0$ and $(\nabla U)^2 > 0$ always
  thus if $\int_\partial \hat{n} \cdot U \nabla U = \int_\partial \hat{n} \cdot U \nabla U = 0$ then $\int_\partial (\nabla U)^2 = 0 \Rightarrow U = 0$ everywhere
  a) Dirichlet boundary condition: $U = 0$ - specify potential $V_1 = V_2$ on boundary
  b) Neuman boundary condition: $\frac{\partial U}{\partial n} = 0$ - specify flux $\frac{\partial U}{\partial n} = \frac{\partial U}{\partial n}$ on boundary

* Continuity boundary conditions - on the interface between two materials

  Flux: $\hat{n} \cdot \mathbf{D} = \varepsilon \mathbf{E}$ (shorthand for now)
  $\mathbf{D} = \int_\partial \hat{n} \cdot \mathbf{D} \cdot d\mathbf{a} = \int_\partial \sigma \cdot d\mathbf{a}$
  $\mathbf{E} = \nabla \times \mathbf{H}$
  $\hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) A = \sigma \cdot A$
  $\hat{n} \times (\mathbf{D}_2 - \mathbf{D}_1) = \sigma$
  $-\frac{\partial \mathbf{D}_2}{\partial n} + \frac{\partial \mathbf{D}_1}{\partial n} = \sigma / \varepsilon_0$

  Flow: $\hat{n} \times \mathbf{E} = \mathbf{H}$
  $\int_\partial \hat{n} \times \mathbf{E} = \int_\partial \mathbf{H}$
  $\hat{n} \times \mathbf{E} = \frac{\nabla \times \mathbf{H}}{\nabla \times \mathbf{E}}$

  the same results obtained by integrating field equations across the normal

  $\nabla \cdot \mathbf{D} = \rho / \varepsilon_0$
  $\nabla \times \mathbf{E} = \mathbf{K}_e S(n)$
  $\nabla \times \mathbf{H} = \mathbf{K}_m S(n)$

  ~ opposite boundary conditions for magnetic fields: $\hat{n} \cdot \mathbf{B} = 0$, $\hat{n} \times \mathbf{H} = \mathbf{K}$