Section 1.1 - Vector Algebra

* Linear (vector) space
  ~ Linear combination: \((a\mathbf{a} + b\mathbf{b})\) is the basic operation
  ~ Basis: \((\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)\) 
    # Basis elements = dimension
  ~ Independence: not collapsed into lower dimension
  ~ Closure: vectors span the entire space

~ Components: \(\mathbf{X} = \sum a_i \mathbf{b}_i = \sum \mathbf{c}_i\) 
  in matrix form:
  \[
  \begin{pmatrix}
  X \\
  Y \\
  Z
  \end{pmatrix}
  =
  \begin{pmatrix}
  a_1 & b_1 & 0 \\
  a_2 & b_2 & 0 \\
  a_3 & b_3 & 0
  \end{pmatrix}
  \begin{pmatrix}
  X \\
  Y \\
  Z
  \end{pmatrix}
  \]
  where \(\mathbf{a} = \hat{\mathbf{a}} \mathbf{a}_1 + \hat{\mathbf{a}} \mathbf{a}_2 + \hat{\mathbf{a}} \mathbf{a}_3 = (\hat{\mathbf{a}} \mathbf{a}_1)(\hat{\mathbf{a}} \mathbf{a}_2)(\hat{\mathbf{a}} \mathbf{a}_3)\) etc

~ Independence: implicit summation over repeated indices

~ Direct sum: \(C = A \oplus B\) add one vector from each independent space to get vector in the product space (not simply union)

~ Projection: the vector \(\mathbf{a} = \mathbf{a}_1 + \mathbf{b}\) has a unique decomposition
  "coordinates" \((a, b)\) in \(A, B\) - relation to basis/components?
  ~ All other structure is added on as multilinear (tensor) extensions

* Metric (inner, dot product) - distance and angle
  \[
  C = \mathbf{a} \cdot \mathbf{b} = \mathbf{a}_1 \mathbf{b}_1 + \mathbf{a}_2 \mathbf{b}_2 + \mathbf{a}_3 \mathbf{b}_3 = a_1 b_1 + a_2 b_2 + a_3 b_3
  \]
  ~ Properties: 1) scalar valued - what is outer product?
    2) Bilinear Form \(a \cdot (b + c) = a \cdot b + a \cdot c\)
    3) Symmetric \(\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}\)

~ Orthonormality and completeness - two fundamental identities help to calculate components, implicitly in above formulas

\[
\begin{pmatrix}
\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 \\
\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3 \\
\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1 \\
\end{pmatrix} = \begin{pmatrix}
\delta_{12} & \delta_{13} & \delta_{13} \\
\delta_{21} & \delta_{23} & \delta_{23} \\
\delta_{31} & \delta_{32} & \delta_{33}
\end{pmatrix}
\]

Kronecker delta: components of the identity matrix

\[
\delta_{ij} = \begin{cases}
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]

~ Orthogonal Projection: a vector \(\mathbf{h}\) divides the space \(X\) into \(X_{\parallel} \oplus X_{\perp}\)

Projection operator: \(P_{\parallel h} = \mathbf{h} \cdot \mathbf{h}^\top\) acts on \(X: \mathbf{h} \cdot \mathbf{y} = \mathbf{h}_\parallel \mathbf{y}\)

~ Generalized Metric: for basis vectors which are not orthonormal, collect all \(n \times n\) dot products into a symmetric matrix (metric tensor)

\[
\mathbf{g}_{ij} = \mathbf{b}_i \cdot \mathbf{b}_j
\]

\[
\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^\top \mathbf{g}_{ij} \mathbf{y}^j = \mathbf{x}^\top \mathbf{b}_i \cdot \mathbf{g}_{ij} \mathbf{y}^j
\]

In the case of a non-orthonormal basis, it is more difficult to find components of a vector, but it can be accomplished using the reciprocal basis (see HW)
Exterior Products - higher-dimensional objects

* cross product (area)
\[ \mathbf{c} = \mathbf{a} \times \mathbf{b} = \hat{n} \mathbf{a} \mathbf{b} \sin \theta = \hat{n} \mathbf{a}_\perp \mathbf{b}_\perp = \mathbf{a}_x \mathbf{a}_y \mathbf{a}_z \mathbf{b}_x \mathbf{b}_y \mathbf{b}_z \]
where \( \hat{n} \perp \mathbf{a} \) and \( \hat{n} \perp \mathbf{b} \) (RH-rule)

\[ d = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \]

~ properties:
1) vector-valued
2) bilinear
3) antisymmetric
\( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \) (oriented)

~ components:
\[ \mathbf{e}_\chi \times \mathbf{e}_\gamma = \mathbf{e}_\zeta^k \mathbf{e}_k \]
where \[ \mathbf{e}_{123} = \mathbf{e}_{231} = \mathbf{e}_{312} = 1 \]
\[ \mathbf{e}_{135} = \mathbf{e}_{351} = \mathbf{e}_{513} = -1 \]

\[ \mathbf{x} \times \mathbf{y} = \mathbf{x}^\chi \mathbf{e}_\chi \times \mathbf{y}^\gamma \mathbf{e}_\gamma = \mathbf{e}_\zeta^k \mathbf{x}^\chi \mathbf{y}^\gamma \mathbf{e}_k \]

~ orthogonal projection: \( \hat{n}_\perp \) projects \( \perp \) to \( \hat{n} \) and rotates by 90°
\[ \hat{x}_\perp = -\hat{n}_\perp (\mathbf{h} \times \mathbf{x}) = \mathbf{P}_\perp \mathbf{x} \quad \mathbf{P}_\perp = -\hat{n}_\perp \hat{n}_\perp = \hat{n}_\perp \hat{n}_\perp \]

\[ \mathbf{P}_\parallel + \mathbf{P}_\perp = \hat{n} \hat{n}^* - \mathbf{n} \times \mathbf{n} = \mathbf{I} \]

~ where is the metric in \( x \)?
vector \( x \) \( \times \) vector = pseudovector
symmetries act more like a 'bivector',
can be defined without metric

* triple product (volume of parallelepiped) - base times height
~ completely antisymmetric - definition of determinant
~ why is the scalar product symmetric \( \times \) vector product antisymmetric?
~ vector \( \times \) vector = pseudoscalar (transformation properties)
~ acts more like a 'trivector' (volume element)
~ again, where is the metric? (not needed!)

* exterior algebra (Grassman, Hamilton, Clifford)
~ extended vector space with basis elements from objects of each dimension
~ pseudo-vectors, scalar separated from normal vectors, scalar
  magnitude, length, area, volume
  scalar, vector, bivector, trivector
\[ \mathbf{\hat{x}}_1, \mathbf{\hat{y}}_1, \mathbf{\hat{z}}_1, \mathbf{\hat{x}}_2, \mathbf{\hat{y}}_2, \mathbf{\hat{z}}_2 \]

~ what about higher-dimensional spaces (like space-time)?
can't form a vector 'cross-product' like in 3-d, but still have exterior product

~ all other products can be broken down into these 8 elements
most important example: BTLC-CAB rule (HW; relation to projectors)
\[ \mathbf{A}_x (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}) \]
\[ \mathbf{E}^i_{jk} A^i (\mathbf{c}^k \mathbf{B}^m \mathbf{c}^n) = (\delta^i_{mj} - \delta^i_{nj} \delta^m_j) A^i \mathbf{B}^m \mathbf{C}^n = \mathbf{B}^i (\mathbf{A} \mathbf{C}) - \mathbf{C}^i (\mathbf{A} \mathbf{B}) \]