* Polar decomposition
  - any matrix $M$ can be expressed as a rotation $R$ and stretch $S$

  - example: shear transformation (see ppt)
  - the stretch $S = V W V^T$ is diagonal after further rotation
  - combine rotations: Singular Value decomposition (SVD)
    $M = R S = R V W W^T = U W V^T$ where $U^T U = I$, $V^T V = I$

* Diagonalization - notation of eigensystems

  $M \tilde{u}_i = \lambda_i \tilde{u}_i$
  $M U = U D$
  $U^T M U = D$

  $U^T U = I$, otherwise $U^T M U = D$

  Similarity transform - change of basis (rotation)

  $M = U D U^T = \sum_i \lambda_i \tilde{u}_i \tilde{u}_i^T$
  $I = U U^T = (\tilde{u}_i \tilde{u}_i^T)(\tilde{u}_j \tilde{u}_j^T) = \sum_i \tilde{u}_i \tilde{u}_i^T$

* Exponential - Normal Matrix analogy.

  - a square matrix $G$ can be decomposed into symmetric $T$ and antisymmetric $A$ parts

    \[
    \frac{1}{2}(G + G^T) = T \\
    \frac{1}{2}(G - G^T) = A
    \]

    \[
    M = T + A
    \]
with respect to the adjoint
- take the exponential of each: \( T^t = T \)  \( A^t = -A \)

a) \( T = VDV^t \)
\( S = e^T = Ve^D V^t \) because \( T^t = T \)

semi positive definite: (+) eigenvalues

b) \( R = e^A \)
\( R^t R = e^{-A} \cdot e^A = e^0 = I \)
unitary
\( \det R = \det e^A = e^{\text{Tr}A} = |e^{i\theta}| = 1 \)

c) \( M = e^G \leftarrow \text{generator} \) if \([T, A] = 0 \) (\( M \) is normal), then
\( = e^{T^t + S} = e^T \cdot e^S = S \cdot R \)

polar decomposition

- summary:
  \( \exp(G = T^t + A) \)
  \( \exp(\omega = \tau + i\psi) \) \( \omega = \psi \)
  \( = (M = S \cdot R) \) \( = (z = s \cdot r) \)
  \( s = e^\tau \) \( r = e^{i\phi} \)

a) Normal \( N = H + iK \)
\( H^t = H \) \( K^t = K \)
\( NN^t = N^t N \) \( HK = KH \) \( \rightarrow n \) complex eigenvalues.

b) (anti) Hermitian \( H^t = H \) \( A^t = (iK)^t = -eK = -A \)

\( \rightarrow \) (imaginary) real eigenvalues

c) Positive definite: positive eigenvalues

c) Unitary: \( U^t U = I \) unit eigenvalues. \( e^{i\theta} \)

- Proofs:
  if \( f(x) = \xi \alpha_i x^i \) and \( M = UDU^t \) then

\( f(M) = \xi \alpha_i M^i = \xi \alpha_i (UDU^t)^i \)
\( = U (\xi \alpha_i D^i) U^t \)
\( = U \cdot \text{diag}(f(x)) \cdot U^{-1} = U \left( \frac{f(x)}{x} \right) U^{-1} \)

\( e^{\text{Tr} M} = \exp(\text{Tr}(U^tDU)) = U \exp(\text{Tr}(\xi \alpha_i x^i)) U^{-1} \)
\[ = U e^{\lambda_1 \hat{u}_1 + \lambda_2 \hat{u}_2 + \cdots} U^{-1} = U \det e^D U^{-1} = \det e^M \]

* Characterization of eigensystems:
- Note: \( (SVD) \): \( M = U W V^T \) but can we say \( M = U D U^{-1} \)?

a) \( \text{End}(n) \): any \( n \times n \) matrix has \( n \) complex eigenvalues however degenerate \( \lambda_i \) may not have have same \( \# \) of eigenvectors
defective eigenvalues \( \approx \) dilation + nilpotent parts of matrix

Example: Shear: \( M = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 10 \\ 0 & 0 \end{pmatrix} + a \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \)
\[ Z(1) = (0) \quad Zv_0 = 0v_0 \quad = I + a Z \]
\[ Z(1) = (1) \quad Zv_1 = v_1 \quad \text{Jordan chain of generalized eigenvectors.} \]
Thus defective matrices still admit Jordan decompositions: \( \det M = U J U^{-1} \)

b) Normal matrices \( N^*N = NN^* \)
equivalently \( N = H + iK \) \( H^T = H \) \( K^T = K \) \( [H,K] = 0 \)
\( H \) and \( K \) each have real eigenvalues
since \( [H,K] = 0 \), there exist common eigenvectors
thus \( N \) has a set of "independent" complex eigenvalues \( N \hat{u}_i = \lambda_i \hat{u}_i \)
\( N^* \) has eigenvalues \( N^* \hat{u}_i = \lambda_i^* \hat{u}_i \)
The eigenvectors are orthogonal \( \hat{u}_i \cdot \hat{u}_j = 0 \) if \( \lambda_i \neq \lambda_j \)
so \( N \) has a unitary diagonalization \( N = U D U^T \) where \( U^T U = I \)

Normal matrix analogy lists the special cases:
Hermitian: real eigs  
Positive def: positive eigs  
Unitary: units $|\lambda_i|=1$

- Proof of simultaneous diagonalization:
  \[ A^* = \lambda \bar{v} \quad AB = BA \]
  Then $A(B^*) = B A \bar{v} = \lambda (B \bar{v})$
  Thus $B \bar{v}$ is an eigenspace $\lambda \bar{v}$ $A \lambda$
  Diagonalize $B$ on $\lambda$, $B \bar{v} = \mu \bar{v}$
  Then $A \bar{v} = \lambda \bar{v}$ also.

- Proof of conjugate eigenvalues & unitary eigenvectors:
  if \( N \bar{v}_i = \lambda_i \bar{v}_i \) and \( N^* \bar{v}_i = \mu_i \bar{v}_i \) then
  \[ u_i^* N^* N u_j = (\lambda_i^* \lambda_j = \mu_i \lambda_j) \quad u_i^* u_j \]
  if $i = j$ then $\lambda_i^* = \lambda_j$
  if $\lambda_i \neq \lambda_j$ then $u_i^* u_j = 0$