* The importance of being ... Earnest?

a) Unitary (orthogonal): simple inversion!
- if $U^t U = I$ then $U^{-1} = U^t$
  eg. component transformations $v' = Uv$ $v = U^t v'$
  similarity (matrix element) transformations $A = UDU^{-1}$
- especially important for infinite-dimensional operators!

b) Hermitian (symmetric): observables!
- if $H^t = H$ then $H = UDU^t$ $D^* = D$ $U^t U = 1$
  real eigenvalues (measurements) & orthogonal eigenstates

c) Diagonal: practical consideration: matrix multiplication
- $A + B = (a_{11} + b_{11}, a_{21} + b_{21}, a_{12} + b_{12}, a_{22} + b_{22}) = B + A$ element-wise addition
  BUT:
  - $AB \neq BA$ in general (non commutative)
  - $f(A) \neq (f(a_{11}), f(a_{21}))$ (non commutative)
  - $(a_{12}, a_{22})$ elements switch around!

  HOWEVER:
  - $A'B' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} = BA'$ for diagonal matrices!
  - if $f(x) = \sum_i f_i x^i$ then $f(A) = \sum_i f_i A^i$
  - so $f(A') = (f(a_{11}), \ldots, f(a_{22}))$
  - if $A = UDU^{-1}$ then $A^2 = UDU^t UDU^{-1} = UDU^t U^{-1}$
\[ f(A) = \sum_{i=0}^{\infty} f_i(U DU^*)^i = U^t(\sum_{i=0}^{\infty} f_i D^i) U = U f(D) U^t \]

d) **Commutative:** \( AB = BA \) **Simultaneous measurements!**

- Physical measurements are represented by the eigenvalues.
- If \( A \) and \( B \) are diagonal, then \( AB = BA \) or the commutator \([A,B] = AB - BA\) equals 0.
- And both \( A,B \) have definite measurements for the basis states \( \hat{e}_1, \hat{e}_2, \ldots \) (canonical basis)
- Is the converse true: if \([A,B] = 0\) then they can be simultaneously diagonalized? \( A = U D_A U^t \) and \( B = U D_B U^t \) for some \( U \) \( \text{YES!} \)
- Thus the commutator is strongly connected to the Heisenberg Uncertainty Principle
  \([\hat{x}, \hat{p}] \psi(x) = i(\hat{p}\psi(x)) - i(\hat{x}\psi(x)) \psi(x) = 0 \Rightarrow p_x \psi \text{ complementarity!} \]

\[ x p_x \text{ product rule:} \]

\[ x^+ \psi \quad \psi^+ x \]

e) **Normal:** \([N, N^+] = 0\) **complex-matrix analogy.**

Let \( H = \frac{1}{2}(N + N^+) \) so \( H^t = H \) and \( N = H + iK \)
\( K = \frac{1}{2i}(N - N^+) \) that \( K^t = K \) and \( N^t = H - iK \)
These are the "real" and "imaginary" parts of \( N \)
Note they both have complex matrix elements!

\[ [N, N^+] = [H + iK, H - iK] = [H, K] - i[H, K] + i[K, H] + [K, K] \]
\[ = -2i[H, K] \text{ so } N, N^+ \text{ commute iff } H, K \text{ do} \]

Then \( H_D = U^t H U = \begin{pmatrix} h_1 & * & \cdots \\ \cdots & \ddots & \cdots \\ * & \cdots & h_n \end{pmatrix} \)
\( K_D = U^t K U = \begin{pmatrix} k_1 & * & \cdots \\ \cdots & \ddots & \cdots \\ * & \cdots & k_n \end{pmatrix} \)

Thus \( D = U^t (H + iK) U = \begin{pmatrix} h_1 + ik_1 & * & \cdots \\ \cdots & \ddots & \cdots \\ * & \cdots & h_n + ik_n \end{pmatrix} = \begin{pmatrix} n_1 & * & \cdots \\ \cdots & \ddots & \cdots \\ * & \cdots & n_n \end{pmatrix} \) or \( N = U D U^t \)
* Classification of normal matrices:

<table>
<thead>
<tr>
<th>$H^t=H$</th>
<th>$n_i \in \mathbb{R}$</th>
<th>Hermitian</th>
<th>$P^t=P$ and $n_i&gt;0$</th>
<th>Positive definite</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^t=T$</td>
<td>$T \in \mathbb{R}^{nn}$</td>
<td>Symmetric</td>
<td>$S^t=S$</td>
<td></td>
</tr>
<tr>
<td>$K^t=-K$</td>
<td>$n_i \in \mathbb{R}$</td>
<td>Anti-Hermitian</td>
<td>$\text{Tr}=e^{i\phi}$</td>
<td>Unitary, $</td>
</tr>
<tr>
<td>$A^t=A$</td>
<td>$n_i \in \mathbb{R}$</td>
<td>Antisymmetric</td>
<td>$\text{Tr}=0$</td>
<td>Orthogonal, $\text{Det}=\pm 1$</td>
</tr>
</tbody>
</table>

* Simultaneous diagonalization theorem:

If $U^tAU=D$ is diagonal and $[A,B]=0$, then $U^tBU$ is block diagonal over the direct sum of eigenspaces of $A$ (with $\lambda_i$).

Proof: Let $A\tilde{u}_i=\lambda\tilde{u}_i$, then $A(B\tilde{u}_i)=BA\tilde{u}_i=\lambda(B\tilde{u}_i)$, thus $B\tilde{u}_i$ is also an eigenvector of $A$ with $\lambda$. Therefore, $B$ maps the eigenspace of $A$ with $\lambda$ into itself.

$$A \begin{pmatrix} \lambda_1 \\ \frac{\lambda_1}{\sqrt{2}} \\ \frac{\lambda_1}{\sqrt{2}} \\ \frac{\lambda_1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \lambda \mathbf{I} \\ \lambda \mathbf{I} \\ \lambda \mathbf{I} \\ \lambda \mathbf{I} \end{pmatrix} \begin{pmatrix} \frac{\lambda_1}{\sqrt{2}} \\ \frac{\lambda_1}{\sqrt{2}} \\ \frac{\lambda_1}{\sqrt{2}} \\ \frac{\lambda_1}{\sqrt{2}} \end{pmatrix}$$

$$B \begin{pmatrix} \lambda_2 \\ \frac{\lambda_2}{\sqrt{2}} \\ \frac{\lambda_2}{\sqrt{2}} \\ \frac{\lambda_2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \frac{\lambda_2}{\sqrt{2}} \\ \frac{\lambda_2}{\sqrt{2}} \\ \frac{\lambda_2}{\sqrt{2}} \end{pmatrix}$$

Note: you can further diagonalize each block $B_i$ without destroying the diagonalization of $A$, since $A_i$ is just a multiple of $I$ and $U^tIU=I$ still.