* Overview: we are going to solve the same problem using the Frobenius method (series solutions). Here are the main steps:

1) dimensionless H: \(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2) \psi = E \psi \Rightarrow \left(\frac{d^2}{dx^2} + \xi^2 - K\right) \psi = 0\)

2) asymptotic form: \(\psi = h(\xi) e^{-\xi^2/2}; \ \xi'' - 2\xi \xi' + (K-1) h = 0\)

3) power series: \(h(\xi) = \sum_{j=0}^{\infty} a_j \xi^j; \ a_{j+2} = \frac{2j+1-K}{(j+1)(j+2)} \)

4) B.C.'s (truncation): \(E_n = \pm \hbar \omega, K_n = \hbar \omega(n+\frac{1}{2})\) \(a_n^{(n)} = -\frac{2(n+1)}{(2j+1)(j+2)}\)

quantization \(\psi_n(\xi) = \sqrt{\frac{\hbar \omega}{2\pi}} e^{-\xi^2/2}; H_n(\xi) \approx e^{-\xi^2/2}\)

* TISE for SHO: \(\hat{A} \psi = E \psi\)

\(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2) \psi(x) = E \psi(x)\)  
(a) solve ODE  
(b) apply B.C.'s.

- dimensionless variables:

let \(\xi = \sqrt{\frac{\hbar \omega}{m}} x; K = \frac{2E}{\hbar \omega} \Rightarrow \frac{d^2}{d\xi^2} + (\xi^2 - K) \psi = 0\)

- asymptotic limit: if \(\xi^2 > K\) then \(\frac{d^2 \psi}{d\xi^2} = -\xi^2 \psi, \ \psi \approx e^{-\xi^2/2}\)

\(\frac{d}{d\xi} \left( \frac{d}{d\xi} e^{-\xi^2/2} \right) = \frac{d}{d\xi} \left( -\xi e^{-\xi^2/2} \right) = -1 \cdot e^{-\xi^2/2} + (-\xi) (-\xi) e^{-\xi^2/2} = \xi^2 e^{-\xi^2/2}\)

let \(\psi(\xi) = h(\xi) e^{-\xi^2/2}\) to factor out this dependence

\(\psi' = h' e^{-\xi^2/2} + h(-\xi) e^{-\xi^2/2}\)

\(\psi'' = h'' e^{-\xi^2/2} + 2h'(-\xi) e^{-\xi^2/2} - h e^{-\xi^2/2} + h \xi e^{-\xi^2/2}\)

\(\psi'' - (\xi^2 - K) \psi = (h'' - 2h' \xi + h \xi^2 + (K-1) h) e^{-\xi^2/2} = 0\)

thus \(\xi'' - 2\xi \xi' + (K-1) h = 0\) Hermite ODE with \(K \xi \rightarrow 2n\)
- Power series solution: let \( h = \sum_{j=0}^{\infty} a_j \xi^j \), \( h' = \sum_{j=0}^{\infty} a_j j \xi^{j-1} \) plus these into ODE:

\[
\frac{d^2 h}{d\xi^2} + \frac{1}{\xi} \frac{dh}{d\xi} + \left( k^2 - \frac{4}{\xi} \right) h = 0
\]

\[
\sum_{j=0}^{\infty} \left( (j+1)(j+2) a_{j+2} - 2j a_j + (k-1) a_j \right) \xi^j = 0
\]

Solution: \( h(\xi) = \left[ h_{\text{ord}} = a_0 + a_2 \xi^2 + a_4 \xi^4 + \ldots \right] + \left[ h_{\text{ord}} = a_0 \xi^2 + a_2 \xi^4 + \ldots \right] \\
\text{Normalization by recursion}
\]

- Quantization: \( a_{j+2} \approx \frac{1}{j+1} a_j \approx \frac{a_0}{(j+1)!} \), exponential growth!

\[
h(\xi) \approx a_0 \xi \frac{\sum_{j=0}^{\infty} + \frac{1}{j!} \xi^j} \xi \approx a_0 e^{\xi^2} \text{ blows up.}
\]

only get normalized solution if series terminates

- if \( k = 2n+1 \) then \( a_{n+2} = a_{n+4} = \ldots = 0 \), \( k_n \) satisfies B.C.

- thus \( E_n = \frac{\hbar \omega}{2} (2n+1) = \hbar \omega (n+\frac{1}{2}) \), \( a^{(n)}_{\text{ord}} = -\frac{2(n-1)}{\xi^{(n+1)}(n+2)} \)

- Hermitian polynomials: "normalized" \( n \) \( \alpha_{n}^{(n)} = 2^n \)

\[
h_0(\xi) = \alpha_0^{(0)} = 1 \quad h_0(\xi) = 1 \quad h_1(\xi) = \alpha_1^{(1)} = 5 \quad h_1(\xi) = 2 \xi \quad H_1(\xi) = 2 \xi
\]

\[
h_2(\xi) = \alpha_2^{(2)} + \alpha_3^{(2)} \xi^2 \approx 1 - 2\xi \quad h_2(\xi) = 2 \xi \quad H_2(\xi) = 4 \xi^2 - 2 \xi
\]

- Normalized wavefunctions:

\[
\psi_n(x) = \frac{\exp(i k_n x)}{\sqrt{2^n n!}} \quad H_n(\xi) e^{-\xi^2/2}
\]

\[
\int_{-\infty}^{\infty} \psi_n(x) \psi_m(x) dx = \sqrt{\pi} \frac{\exp(-\xi^2)}{2^n n!} \int_{-\infty}^{\infty} H_n(\xi) H_m(\xi) = \delta_{nm}
\]

- Classical probability density:

\[
p dx = \text{probability that } x \in (x,t) dx\text{ at time } t
\]

\[
x(t) = x_0 \sin wt
\]
Thus $\rho(x) \propto \frac{1}{x^2} = \frac{1}{\cos(\varpi t)} = \frac{1}{\sqrt{1-\sin^2 \varpi t}} = \frac{1}{\sqrt{1-u^2}} \int_0^x k \, dx = k x_0 \int_0^1 \frac{du}{\sqrt{1-u^2}} = k x_0 \frac{\pi}{2} = 1 \quad u = \frac{x}{x_0}$

thus $\rho(x) = \frac{2\pi}{\sqrt{x_0^2 - x^2}}$

Compare with $|\psi(x)|^2$ to see classical limit.

(Griffiths)

Figure 2.5: (a) The first four stationary states of the harmonic oscillator. (b) Graph of $|\psi(x)|^2$, with the classical distribution (dashed curves) superimposed.