Lecture Notes #02Θ — *Thu 17 Jan 2002*

Superposition of *F*orces & *E*-Fields Vectors

Superposition of Forces and Electric Fields

A fundamental feature of forces in the Newtonian Universe is expressed through the *Principle of Superposition*: ***The total force*** $\boldsymbol{F}_{\rm net}$ ***on any single object equals the sum of all the individual forces acting on that object.*** "Sum" of course means ***vector sum*** here.

Recall that, according to Coulomb's Law, the electrostatic force $\boldsymbol{F}_i$ on an object of net charge q due to any other object of charge Q_i is

$$\boldsymbol{F}_i = k\,\frac{Q_i q}{r_i^2}\,\hat{\boldsymbol{r}}_i$$

where the distance between them is r_i and the direction from Q_i to q is $\hat{\boldsymbol{r}}_i$. (We have also assumed, as usual, that we can pretend that the charges q and Q_i live on essentially *point* objects—much smaller in size than r_i.) If any number of objects $\#i = 1, \ldots, N$ exert electrostatic forces on q, then *every one of the Coulomb forces* $\boldsymbol{F}_i$ *is proportional to* q. Hence the net force on the charge is also proportional to its value q:

$$\boldsymbol{F}_{\rm net} = \sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{i=1}^{N} k\,\frac{Q_i q}{r_i^2}\,\hat{\boldsymbol{r}}_i = q \times \sum_{i=1}^{N} k\,\frac{Q_i}{r_i^2}\,\hat{\boldsymbol{r}}_i \equiv q \times \underbrace{\boldsymbol{E}(\text{at } q)}_{\text{"Electric Field" at position of } q}\ .$$

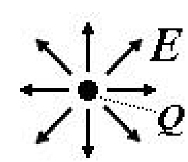
The value of the electric field at q's location is simply *defined* to be the vector $\boldsymbol{F}_{\rm net}/q$.

The "electric field" at any point always refers to the **net** field at that point—the sum of all charges' individual contributions of the quantity $\hat{\boldsymbol{r}}_i k Q_i / r_i^2$ to the total vector $\boldsymbol{E}$ at that point. ***Electric fields obeys the Superposition Principle*** just as forces do.

This is a more profound statement than it might at first appear. It tells us that, no matter how complicated the distribution and strengths of "source" charges in a region of space, their total effect on any other "test" charge placed *any*where in that region follows from knowing how they affect the test charge *one source charge at a time*. And this we can always get from the Coulombic force between a *pair* of charges. So for one or two source charges, or a triangle of charges, or a stick of charge (made of lots of little point charges), or a flat disk of charge ("), or a ball of charge ("), or even a cloud of charges shaped like Albert Einstein (")—the electric field at any point of interest around all the charges $\#i = 1, \ldots, N$ can, for each situation, be obtained by calculating the $\sum_{i=1}^{N}$ of all the source charges' vectors $\hat{\boldsymbol{r}}_i k Q_i / r_i^2$ evaluated at that point. It might be technically complicated to carry out the Σum,* but in principle that's all you have to do.

What really interests us, however, is not the calculation *per se* but the ***pattern*** of the electric field around various collections of charges—pair, triangle, stick, disk, ball, Albert, &c. Picture the collection as a snapshot of a swarm of N bees, where $N \geq 1$ and perhaps $N =$ lots, each bee representing one charge. We already have a few clues about how the electric-field pattern must appear in the *limits of "very close" and "very far"*:

- If we penetrate such a swarm of charges and get very, very close to any one of them, we can tune the electric-field contribution from that single charged bee to arbitrarily large magnitudes by getting even closer to it ($r \to 0 \Rightarrow |\boldsymbol{E}| \propto 1/r^2 \to \infty$)—as if that one's buzzing wings totally dominated our field of view.



* If we are given the location of all the point charges, then picking an "observation point" to measure $\boldsymbol{E}$ suffices to determine all the distances $\{r_i\}$ to and directions $\{\hat{\boldsymbol{r}}_i\}$ from the source charges $\{Q_i\}$. And if we place a charge q at that point we then know the net force on it.

- On the other hand, imagine that we recede to a large enough distance r from the whole swarm, until it looks like just one blobby charge *over there*. Then we ought to see a net field pretty similar to the elemental Coulomb's-Law field $\hat{\boldsymbol{r}}kQr^{-2}$ due to a *single charge*. The value of that charge Q would of course be just the total charge of the swarm: $Q = Q_{\text{total}} = \sum_{i=1}^{N} Q_i$. In other words, no matter what the shape of the swarm, very far away from it we can hardly discern its shape and we only see a nearly-pointlike charge Q_{total}. (This is true so long as all the charge resides in a *limited* region of space. If it's spread out over such a large region that we can't really "get away" from it, then its effect might never completely "go away." We'll see examples of this next time via Gauss's Law.)

In particular, a *neutral* collection of charges ($\sum_{i=1}^{N} Q_i = 0$) always produces an $\boldsymbol{E}$-field at large distance that might be characterized by Coulomb's Law with ZERO charge. But what's interesting is that this does NOT mean that the electric field exactly equals ZERO far away. How could it? Coulomb's Law provides no abrupt cut-offs, no sudden "switch-offs" from nonzero to zero forces. Rather, the field $\boldsymbol{E}$ far from a neutral object—and hence the force $\boldsymbol{F} = q\boldsymbol{E}$ on any charge q sitting out there—wanes with increasing distance, as expected, but *faster than the r^{-2}-dependence* of a charged particle. For example, far away from an electric *dipole* $\boxed{+-}$ the $\boldsymbol{E}$-field strength (i.e., the force) dies off in any direction in proportion to $r^{-3} = 1/r^3$ rather than $1/r^2$. And for a square of alternating charges like $\boxed{\begin{smallmatrix}+&-\\-&+\end{smallmatrix}}$ (called an electric *quadrupole*), the field dies off even more rapidly with distance, in proportion to $1/r^4$. And so on.

In the next LECTURE, we will see how particular swarm-shapes of charge—long sticks, large flat plates, and spheres—produce specific kinds of electric-field patterns.

An Example of Superposition—Giancoli/5, Problem 16–33

33. (II) (*a*) Determine the electric field **E** at the origin *O* in Fig. 16–42 due to the two charges at *A* and *B*. (*b*) Repeat, but let the charge at *B* be reversed in sign.

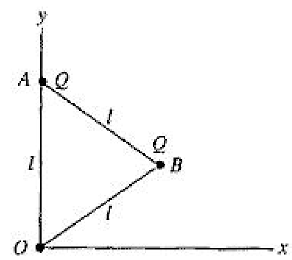


FIGURE 16–42 Problem 33.

Here's an Example of superposition of $\boldsymbol{E}$-fields—the calculation of the net $\boldsymbol{E}$-field from 2 charges (Q and $\pm Q$) at a point that happens to lie at the third vertex O of an equilateral triangle formed with the two charges' positions. You might foresee that the third corner of such a triangle lies on the bisector of the line connecting the two charges, and that the problem has a symmetry with respect to the two sides of the bisector. The field anywhere on this bisector should consequently have a special direction, depending on whether the two charges have the same or opposite sign.

(a) *N.B.* We have to *assume* that $Q > 0$ to be able to answer this problem. (How would the answers change if it turned out that $Q < 0$?)

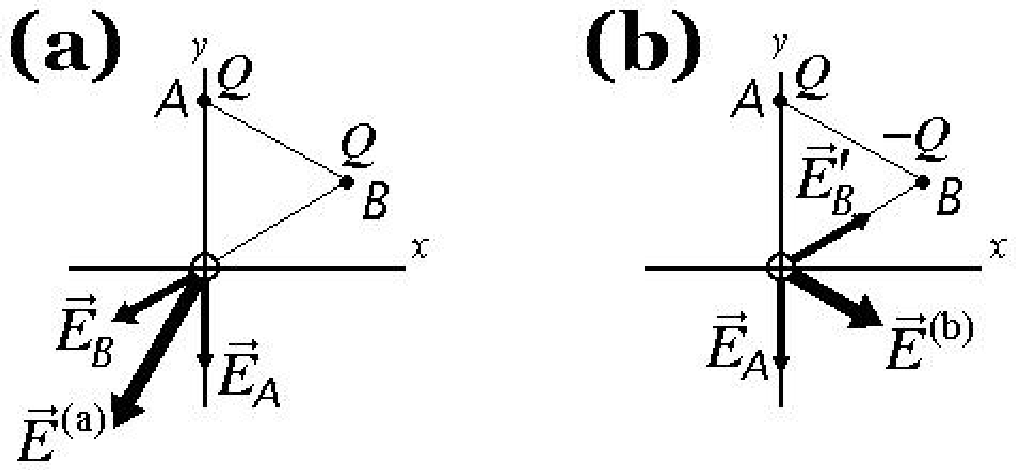
The origin O is the same distance l from each of the two charges at A and B. And since these charges both equal Q, the magnitude of the electric fields due to both are the same at O:

$$|\boldsymbol{E}_A| = |\boldsymbol{E}_B| = k\,\frac{|Q|}{l^2} = \frac{1}{4\pi\epsilon_0}\,\frac{|Q|}{l^2}\;.$$

(*Reminder:* You should write vectors such as $\boldsymbol{E}_A$ and $\boldsymbol{E}_B$ like "$\vec{E}_A$" and "$\vec{E}_B$".)

Always try to picture or guess a problem's answer, at least very roughly, before actually clawing through the details.

The two vectors are shown below along with a thicker vector $\boldsymbol{E}^{(a)}$ indicating their total [the superscript just signifies part "(a)" of this problem]. Since $\boldsymbol{E}_A$ points straight down (i.e., along $-\hat{\boldsymbol{y}}$) and $\boldsymbol{E}_B$ points down and to the left (between $-\hat{\boldsymbol{x}}$ and $-\hat{\boldsymbol{y}}$), the result $\boldsymbol{E}^{(a)}$ ought to point more down and to the left.



As to the detailed calculation of $\boldsymbol{E}^{(a)}$, we need the x- and y-components of both vectors. There is no x-component to $\boldsymbol{E}_A$, so $E_{Ax} = 0$ [$\cos(-90°) = 0$]. Then the direction from the charge at A to the observation point O is $\parallel -\hat{\boldsymbol{y}}$, so $E_{Ay} = -|\boldsymbol{E}_A|$ [$\sin(-90°) = -1$]. The equilateralness of the triangle implies that the line OB makes an angle of 30° with the x-axis. So the direction from the charge at B to O is in the opposite direction, which means 30° *below* the $-x$-axis on the left side of the y-axis. Then we have the components $E_{Bx} = -|\boldsymbol{E}_B|\cos 30°$ and $E_{By} = -|\boldsymbol{E}_B|\sin 30°$.

Now add the components of $\boldsymbol{E}_A$ and $\boldsymbol{E}_B$ to get the components of the total vector:

$$
\begin{aligned}
\boldsymbol{E}^{(a)} &= \boldsymbol{E}_A + \boldsymbol{E}_B \\
&= \quad 0\,\hat{\boldsymbol{x}} \quad - \quad |\boldsymbol{E}_A|\,\hat{\boldsymbol{y}} && \Longleftarrow \quad \boldsymbol{E}_A \\
&\quad - |\boldsymbol{E}_B|\underbrace{\cos 30°}_{\sqrt{3}/2}\hat{\boldsymbol{x}} \quad - \quad |\boldsymbol{E}_B|\underbrace{\sin 30°}_{1/2}\hat{\boldsymbol{y}} && \Longleftarrow \quad \boldsymbol{E}_B \\
&= -\frac{\sqrt{3}}{2}\frac{kQ}{l^2}\hat{\boldsymbol{x}} \quad - \quad \frac{3}{2}\frac{kQ}{l^2}\hat{\boldsymbol{y}}
\end{aligned}
$$

(When we added the y-components, we came up with $-|\boldsymbol{E}_A| - \frac{1}{2}|\boldsymbol{E}_B| = -\frac{3}{2}|\boldsymbol{E}_B|$ since $|\boldsymbol{E}_A| = |\boldsymbol{E}_B|$.) So the answer is

$$\boxed{\boldsymbol{E}^{(a)} = \frac{kQ}{l^2}\left(-\frac{\sqrt{3}}{2}\,\hat{\boldsymbol{x}} - \frac{3}{2}\,\hat{\boldsymbol{y}}\right)}$$

It's instructive to note that the *magnitude* of this total is

$$\begin{aligned}\left|\boldsymbol{E}^{(a)}\right| &= \sqrt{\left[E_x^{(a)}\right]^2 + \left[E_y^{(a)}\right]^2} \\ &= \frac{kQ}{l^2}\sqrt{\left[-\frac{\sqrt{3}}{2}\right]^2 + \left[-\frac{3}{2}\right]^2 = \frac{3}{4} + \frac{9}{4} = 3} \\ &= \sqrt{3}\,\frac{kQ}{l^2}\,.\end{aligned}$$

And if we want to rewrite the above $\boxed{\boldsymbol{E}^{(a)} = \cdots(\cdots)}$ with this magnitude indicated explicitly, we have to factor out $\sqrt{3}$ from the "vector part" $[\cdots]$, which yields

$$\boxed{\boldsymbol{E}^{(a)} = \sqrt{3}\,\frac{kQ}{l^2}\left[-\frac{1}{2}\,\hat{\boldsymbol{x}} - \frac{\sqrt{3}}{2}\,\hat{\boldsymbol{y}}\right]}$$

The point† is that since $|\boldsymbol{E}^{(a)}| = |\sqrt{3}kQ/l^2|$, the remaining vector factor $[\cdots]$ here must have magnitude: $|[\cdots]| = 1$. Hence it is *the unit vector* parallel to $\boldsymbol{E}^{(a)}$ (we might call it $\hat{\boldsymbol{r}}^{(a)}$), indicating the direction of $\boldsymbol{E}^{(a)}$. Check that it correctly points 60° below the x-axis to the left of the y-axis!

† $\boldsymbol{E}^{(a)}$ has been rewritten here in the same spirit as we might write $-5 = [-1]\times|-5| = [-1]\times 5$, where $|-5| = 5$ is the *magnitude* of the quantity—always zero or positive—and the *sign* $[-1]$ is its unit-magnitude "direction" (actually called the number's *phase*).

(b) This version, with a sign-reversed charge $-Q$ at point B, is essentially the same as the first part with $+Q$ at point B. The field $\boldsymbol{E}_A$ is precisely the same as before. The new field $\boldsymbol{E}'_B = -\boldsymbol{E}_B$ from the charge at B has the same magnitude as before ($|\boldsymbol{E}'_B| = |\boldsymbol{E}_B|$) but now has its direction reversed—simply because the charge responsible would now be attractive rather than repulsive on a positive test charge. (*Remember*: The **E**-field at a point is practically the same thing as the *force* that would be experienced by a *positive* test charge placed at that point.)

Adding the components of the fields produced at point O by the charges Q at A and $-Q$ at B, we have

$$\begin{aligned}
\boldsymbol{E}^{(b)} &= \boldsymbol{E}_A + \boldsymbol{E}'_B \\
&= \quad 0\,\hat{\boldsymbol{x}} \quad - \quad |\boldsymbol{E}_A|\,\hat{\boldsymbol{y}} \quad = \quad \boldsymbol{E}_A \\
&\quad + |\boldsymbol{E}'_B|\underbrace{\cos 30^\circ}_{\sqrt{3}/2}\hat{\boldsymbol{x}} \quad + \quad |\boldsymbol{E}'_B|\underbrace{\sin 30^\circ}_{1/2}\hat{\boldsymbol{y}} \quad = \quad \boldsymbol{E}'_B \\
&= \quad \frac{\sqrt{3}}{2}\,\frac{kQ}{l^2}\,\hat{\boldsymbol{x}} \quad - \quad \frac{1}{2}\,\frac{kQ}{l^2}\,\hat{\boldsymbol{y}}
\end{aligned}$$

$$\Longrightarrow \boxed{\boldsymbol{E}^{(b)} = \frac{kQ}{l^2}\left[+\frac{\sqrt{3}}{2}\,\hat{\boldsymbol{x}} - \frac{1}{2}\,\hat{\boldsymbol{y}}\right]}$$

This time the magnitude of the result reduces to just

$$\left|\boldsymbol{E}^{(b)}\right| = \frac{kQ}{l^2}$$

—WORK IT OUT—because the "vector part" $\left[\frac{\sqrt{3}}{2}\hat{\boldsymbol{x}} - \frac{1}{2}\hat{\boldsymbol{y}}\right]$ already happens to be a unit vector—CHECK THAT TOO, AND CHECK THAT THE DIRECTION IS CORRECT!