Lecture Notes #03Θ — *Thu 24 Jan 2002*

Electric Potential, Potential Differences, & Potential Energy

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We have thus far talked about electrostatic *forces*, all of which ultimately arise from Coulomb's Law. All ES forces on an arbitrary "test" charge q—forces produced by any number of other "source charges"—are proportional to q. Therefore, the net force $\boldsymbol{F}_{\text{net}}$ on q is always proportional to the value q. (E.g., if you double q you double all the forces on it; if you change the sign of q, you reverse the direction of all the forces on it, &c.) And so it has made sense for us to define a "force-per-q" vector $\boldsymbol{E} = \boldsymbol{F}_{\text{net}}/q$ at every point—the ***Electric Field***—in order to characterize the virtual effect of the source charges at any point P in space. What's the actual "effect"? It's the ***Force*** on any charge q that is placed at P: a force $\boldsymbol{F}_{\text{net}}(\text{at P}) = q \times \boldsymbol{E}(\text{at P})$ acts on the q placed there, whether $q > 0$ or $q < 0$. And of course where there are forces there is motion and acceleration—observable stuff.

That's all about forces. But what about ***Energy***?

In parallel to the $\boxed{\boldsymbol{E} \equiv \boldsymbol{F}_{\text{net}}/q}$ idea, we will define a quantity called the ***Electric Potential*** V that is related to the ***Electric Potential Energy*** [PE] of a charge q by $\boxed{V \equiv \text{PE}/q}$. And just as the electric field $\boldsymbol{E}$ characterizes only the possible effects of source charges (not q) at different points in space, so will the potential depend only on work the sources *could* perform (but not on the value of any particular "test" charge). In contrast, just as $\boldsymbol{F}_{\text{net}}$ is an actual force on q exerted by the source charges, PE is an actual energy of some q arising from its interaction with the source charges.

KEEP THE FOLLOWING IN MIND: An electric field $\boldsymbol{E}$ represents "virtual effects," "messages," "potentialities" characteristic of a set of charges $\{Q_i\}$ in space—but ***not a complete interaction***! *Only when some/any other* *charge q is brought into that environment to "experience," "feel," "test," or otherwise "sniff around" for other charges does something actually happen*—namely, a force $\boldsymbol{F}_{\text{net}} = q\boldsymbol{E}$ on the charge q. So, too, is the electric "potential" V at a point a measure of the virtual energetics characteristic of a set of charges $\{Q_i\}$—while the electric "*potential energy*" PE $= qV$ is the *actual energy expended* in moving a real charge q to that point. THE MECHANICAL QUANTITIES $\boxed{\boldsymbol{F} \text{ \& PE}}$ ALWAYS INVOLVE PRODUCTS OF PAIRS OF CHARGES LIKE $\underline{Q_i q}$: Message Sent (Q_i) & Received (q). The "job-half-done" quantities $\boldsymbol{E}$ and V instead always involve all sources Q_i but never the "test" charge q: Message Sent but nothing has received it yet.

The Role of Energy

Recall that in Newtonian Mechanics forces serve to accelerate masses, thereby changing their velocities, pushing them around, rearranging objects in general, and *redistributing their energy* amongst them. **Energy** is widely regarded as the single most powerful and unifying concept in physics. Here are some features of **Energy**:

- **Energy** comes in various interchangeable forms: kinetic energy ($\frac{1}{2}mv^2$), chemical energy, rest-mass energy ($E = mc^2$), heat, stored potential energy [PE] in a spring ($\frac{1}{2}k[\Delta x]^2$), gravitational PE ($mg[\Delta h]$), *electric* PE (. . . ? . . .), . . .
- **Energy** is a *scalar* quantity. That is, it is just a number, *not a vector!* It has no direction, just a value. The value might vary from point to point, as it often does with PE. And one can add several contributions to a particle's PE at any one point . . . but that's just adding numbers (not nasty vectors).
- *Potential Energy* refers to **Energy** that is somehow "stored" in a system. For "conservative" forces such as the Newtonian gravitational force and the ES (Coulombic) force, you can always "recover all the stored energy" (at least in principle). If you shoot

an arrow into the air, the kinetic energy you impart to it becomes converted into gravitational PE as it reaches the peak of its trajectory (where it moves most slowly), then this PE is reconverted back into kinetic energy as it speeds up and falls back to Earth (tho' you know not where). Friction is not a conservative force—it refers to energy that gets irreversibly dispersed—but neither is it a *fundamental force* like gravity and Coulombic forces.

- Physically measurable effects arise *only from energy differences—EVER!* So ONLY DIFFERENCES IN POTENTIAL ENERGY EVER MATTER. For a given situation or problem or particle, one can choose *any* point and identify its $\text{PE} \equiv \text{PE}_0$ there as a reference value, and then only relative amounts above or below PE_0 are important. For example, some typical reference positions are: for stored PE of a mass hanging on a spring—its equilibrium position; for gravitational PE—the altitude "mean sea level" ... or perhaps the top of Mt. Everest. For the ES potential energy among point charges, one usually chooses ZERO potential energy to correspond to positions somewhere way off towards ∞. Note that if it takes the same amount of work to haul a particle over to any point in a certain region of space, then the particle has the same PE in that whole region—which would be a kind of plateau—and this value might as well be called ZERO. In other words, the PE can be shifted by whatever amount is convenient to the situation.
- Only energy differences ever matter because, in fact, interactions always involve an *exchange* of energy. It's a "zero-sum" game.
- Hence, we have CONSERVATION OF ENERGY:

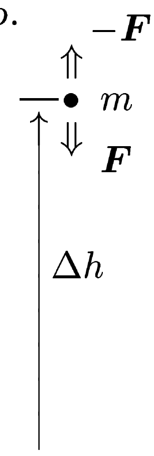
The total energy in a closed system is always conserved.

The UNIVERSE itself is just such a *closed system*—whose total energy doesn't vary. ... Conservation laws hold a very special pride of place in modern physics. In this course, CONSERVATION OF [TOTAL] ENERGY and CONSERVATION OF [TOTAL] CHARGE are the most fundamental facts constraining what may happen in our the electric Universe. (These laws'll pop up again soon enough—as Kirchhoff's Rules for analyzing DC circuits.)

Relating Forces to Energy

Going "uphill"—against a force—increases potential energy.

Consider the work YOU do by exerting a force $\boldsymbol{F}_{\text{yours}}$ to drag a particle at constant speed *against* a force $\boldsymbol{F}$ that is constant. For example, drag a mass m upwards against the downward gravitational force near the Earth's surface. Since $\boldsymbol{F} = \boldsymbol{F}_{\text{grav}}$ points *down*, in this case, $\boldsymbol{F}_{\text{yours}} = -\boldsymbol{F}$ points *up*.



YOU do positive work $W_{\text{by you}} = |\boldsymbol{F}_{\text{yours}}| \times \Delta h$ to lift m, where Δh is the change in altitude = distance traveled *up* ($\| -\boldsymbol{F}$).

$\Longrightarrow$ This work is **Energy** *added* to the "store" of the system's PE.

Since $\Delta h \gtrless 0$ goes $\left\{ {\text{up} \atop \text{down}} \right\}$, then $W_{\text{by you}} \gtrless 0$ and $\Delta\text{PE} \gtrless 0$.
(Moving horizontally, along paths $\perp \boldsymbol{F}$, requires no energy.)

So for lifting a mass m directly upwards,

$$W_{\text{by you}} = F_{\text{yours}} \times \Delta h = |F_{\text{grav}}| \times \Delta h = mg \times \Delta h \equiv \Delta\text{PE}\,.$$

To get the sign to come out correctly for a one-dimensional displacement Δh in the presence of a nonzero force $\boldsymbol{F}$ in one dimension, we should use

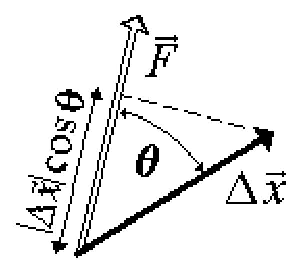
$$\boxed{W_{\text{by you}} = F_{\text{yours}} \times \Delta h = -F \times \Delta h \equiv \Delta\text{PE}}\,.$$

E.g., if we take "up" as the positive direction (towards increasing altitude h) the downwards force of gravity is $F = -mg < 0$ and

$$W_{\text{by you}} = F \times \Delta h = -(-mg) \times \Delta h = mg\Delta h \gtrless 0 \text{ if } \Delta h \gtrless 0\,,$$

which works out as it should. By the way, the work done *by you* is the OPPOSITE of the work done *by gravity*: $W_{\text{by you}} = -W_{\text{by grav}}$, since the forces $\boldsymbol{F}_{\text{yours}}$ & $\boldsymbol{F}_{\text{grav}}$ are opposite. E.g., if you move something downwards, gravity does positive work (its downwards force $F < 0$ is then parallel to the displacement $\Delta h < 0$) while you do negative work—you gain energy in this case rather than give it away. **Energy** is always a zero-sum game: the energy lost by one participant is always gained by another, and *vice-versa. Note:* By moving things at *constant speed* we can ignore the KE since $\Delta\text{KE} = 0$.

For gravity near the Earth, the potential energy *per mass*, $W_{\text{by you}}/m = g\,\Delta h$, is a property of the source producing the gravitational field (Earth) and not of the "test" mass m. We call this quantity the "*potential*" of the gravitational field.



Reverting to more than one dimension, the change of position Δh need not be parallel to $\boldsymbol{F}_{\text{yours}} = -\boldsymbol{F}$ ("up," for $W_{\text{by you}} > 0$) or to $+\boldsymbol{F}$ ("down," for $W_{\text{by you}} < 0$). The vector displacement $\Delta\boldsymbol{x}$ would then make some angle θ with the vector $\boldsymbol{F}$. The relevant part of the displacement for doing work is only the part of $\Delta\boldsymbol{x}$ that lies *along* $\boldsymbol{F}$, which is $|\Delta\boldsymbol{x}|\cos\theta$ (notice the magnitude $|\Delta\boldsymbol{x}|$), so

$$W_{\text{by you}} = -|\boldsymbol{F}| \times (|\Delta\boldsymbol{x}|\cos\theta) \quad \text{or} \quad \boxed{W_{\text{by you}} = -|\boldsymbol{F}|\,|\Delta\boldsymbol{x}|\,\cos\theta \equiv -\boldsymbol{F}\cdot\Delta\boldsymbol{x}}$$

The "dot product" $\boldsymbol{F}\cdot\Delta\boldsymbol{x}$ is just a shorthand for taking the *two vectors* $\boldsymbol{F}$ and $\Delta\boldsymbol{x}$ and producing the *scalar* (number) "$W_{\text{by you}}$". Note that once again this correctly reduces to $W_{\text{by you}} > 0$ when $\boldsymbol{F}$ and $\Delta\boldsymbol{x}$ are *anti*parallel to one another ($\theta = 180°$) and to $W_{\text{by you}} < 0$ when $\boldsymbol{F} \parallel \Delta\boldsymbol{x}$ ($\theta = 0°$). Also, the work is correctly ZERO when $\boldsymbol{F} \perp \Delta\boldsymbol{x}$ ($\theta = 90°$): force perpendicular to motion cannot do any work (no speeding up or slowing down occurs), though it might change the *direction* of motion.

You may have expected the expression for the work done by you to involve a plus sign: $W_{\text{by you}} = +|\boldsymbol{F}|\,|\Delta\boldsymbol{x}|\,\cos\theta$ rather than $-|\boldsymbol{F}|\,|\Delta\boldsymbol{x}|\,\cos\theta$.

It would, were we talking about the work done by the force $\boldsymbol{F}$—but we're not! The minus sign is there because we are referring not to energy expended by the system (i.e., work done by $\boldsymbol{F}$) but rather energy *added to* the system by us outside agents via $\boldsymbol{F}_{\text{yours}} = -\boldsymbol{F}$, which is the OPPOSITE thing.

Let's summarize the interplay between energy expenditure (lifting or climbing) and potential (vertical position) metaphorically:

> **You "climb up" to a "higher" potential.**
> **You "fall down" to a "lower" potential.**

This always holds for masses and gravity. *It also holds for positive charges in ES*: You gain PE by climbing to higher potentials, lose PE by falling to lower potentials. Negative charges, as usual, have the OPPOSITE behavior: they "fall up" towards higher potentials and lose PE; they gain PE by "climbing down" to lower potentials.

In any case, your intuition is probably the best guide to your remembering how PE changes: When you expend a positive amount of energy, i.e., when you perform positive work on the charges to make something happen to them—anything—then the charges' PE INcreases. And when you exert negative energy, i.e., when the charges perform positive work *on you*, then the charges' PE DEcreases ($\Rightarrow$ and $\Leftarrow$ below are forces that *you exert*):

$\Rightarrow \boxed{+} \quad \boxed{+} \Leftarrow$	$\Leftarrow \boxed{+} \quad \boxed{+} \Rightarrow$	$\Rightarrow \boxed{+} \quad \boxed{-} \Leftarrow$	$\Leftarrow \boxed{+} \quad \boxed{-} \Rightarrow$
INCREASE PE	DECREASE PE	DECREASE PE	INCREASE PE

Electric Potential V (*vs.* Potential Difference ΔV)

Compare	FORCE	*vs.*	**Energy** (WORK)
Compare	$\frac{\boldsymbol{F}}{q}$	*vs.*	$\frac{W}{q}$
Compare	$\boldsymbol{E}$	*vs.*	ΔV

In a region where you work against a force $\boldsymbol{F}$ that is CONSTANT (at least approximately), then the work $W_{\text{by you}} = -\{\texttt{Force}\} \times \{\texttt{distance}\}$ increases in proportion to the displacement: the further you push against the force, the more work you do. (It's a *linear* dependence, $W_{\text{by you}} \propto \Delta x$.) Then using $\boldsymbol{F} = q\boldsymbol{E}$ as the "$\{\texttt{Force}\}$" shows that

$$\boxed{\Delta V \equiv \frac{W_{\text{by you}}}{q} = \frac{-\boldsymbol{F}\cdot\Delta\boldsymbol{x}}{q} = -\boldsymbol{E}\cdot\Delta\boldsymbol{x}}$$

($\boldsymbol{E}$ and $\Delta\boldsymbol{x}$ are vectors, whereas ΔV is a number). This simply means that as WE do work to move a charge around against ES forces, the *Work we do per unit charge moved* ($\Delta V = -\boldsymbol{E}\cdot\Delta\boldsymbol{x}$) equals the **opposite** of the *Work the field does per unit charge moved* ($[\boldsymbol{F}\cdot\Delta\boldsymbol{x}]/q = +\boldsymbol{E}\cdot\Delta\boldsymbol{x}$).

This is just CONSERVATION OF ENERGY!

If $\Delta V = V_b - V_a \equiv \Delta V_{ba}$ is the potential difference between any two points a and b in space, the *work we do* on any charge q in moving it from a to b is just $W_{\text{by you}} = q\,V_b - q\,V_a = q\,\Delta V_{ba}$. This *work we do* gets "stored" as extra potential energy so, in general,

$$\boxed{\Delta\text{PE} = W_{\text{by you}} = q\,\Delta V}\,.$$

If we climb "uphill," say, approaching a positive source charge, then $\Delta V > 0$. And then the potential **Energy** of a positive test charge q increases ($q\,\Delta V > 0$), as it should if we are shoving it towards a positive source charge. If we take the same uphill climb ($\Delta V > 0$) dragging instead a *negative* charge q, its potential *energy* DEcreases ($q\,\Delta V < 0$)—even though we are going to a higher potential ($\Delta V > 0$)—as it should if we are being attracted by a negative charge ($q < 0$) towards a positive source charge.

And if we approach a source charge that is negative ... we go "downhill" towards it and the potential decreases ($\Delta V < 0$). A positive q's PE would now decrease too (as it gets attracted and falls downhill towards the negative source charge, $q\,\Delta V < 0$), while a negative q's PE would in fact increase (as we try to shove two like charges together, $q\,\Delta V > 0$).

The potential energy between two charges with the same (opposite) sign should increase (decrease) as they approach one another ... because it would take positive (negative) work on our part to bring them together. For the simplest case, one source point charge Q_1, the *potential* at a distance r from it turns out to equal

$$V(r) = \frac{kQ_1}{r}$$

(we have chosen the reference value $V = 0$—"sea level"—to hold at $r \to \infty$). The *potential difference* between any two points at distances r_a and r_b is

$$\Delta V_{ba} = V_b - V_a = V(r_b) - V(r_a) = \frac{kQ_1}{r_b} - \frac{kQ_1}{r_a}\,.$$

Then the *potential energy*—the energy to move another charge Q_2 from a to b is

$$\Delta\text{PE} = W_{ba} = Q_2\,\Delta V_{ba} = \frac{kQ_1Q_2}{r_b} - \frac{kQ_1Q_2}{r_a}$$

... And if the initial point lies extremely far away ($r_a \to \infty$), the *potential energy between two charges separated by a distance* $r = r_b$ is just *the work you would have to do to bring them together from a great distance*:

$$\boxed{\mathrm{PE}_{12}(r) = \frac{kQ_1Q_2}{r}}$$

Note how you can't tell from this **Energy** [CHECK THE UNITS!] which charge is the "source" and which is the "test." There is no difference really; there are simply two interacting charges that we label in some way or other. (The same symmetry holds between the two interacting charges in Coulomb's force law, remember? $\boldsymbol{F} = \hat{\boldsymbol{r}}\, kQ_1Q_2/r^2$.)

The closer we try to cram together two charges of the same sign ($Q_1Q_2 > 0$, decreasing r), the more the energy cost PE_{12} increases in value. Similarly, the further we try to yank apart two charges of opposite sign ($Q_1Q_2 < 0$, increasing r), the more the energy cost PE_{12} increases—it's negative but increases towards zero. "Increasing energy cost" means the system gains and stores the energy we've expended and lost, the work we've done. Alternatively, the more we try to pull apart two charges of the same sign or push together two charges of opposite sign, the lower the energy cost: PE_{12} then *decreases*. This means that the charges spend some of their potential energy to push and pull and do work *on us*, rather than *vice-versa*; then we gain whatever energy is lost by the system of charges.

By the way, ANY two points with $r_a = r_b$—at the same distance from Q_1—have the same value of the potential. These points do NOT have to lie along a straight line through the point charge; only the radial distance from Q_1 matters. In such cases, when the potential difference equals ZERO ($\Delta V_{ba} = 0$), it costs NO ENERGY to move any charge q whatsoever from a to b, since $\Delta\mathrm{PE} = W_{ba} = q\,\Delta V_{ba} = 0$. It would be like following a *contour line* around a hill at constant altitude—no energy is spent or gained since no climbing uphill or downhill is involved.