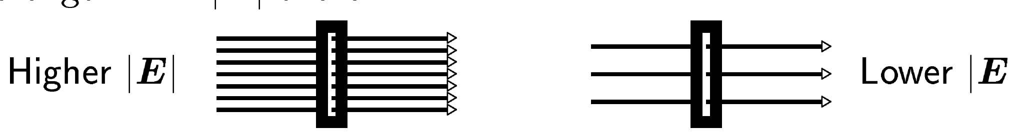
# Lecture Notes #03T — *Tue 22 Jan 2002*

## Gauss's Law & Electric Fields For Some Special Charge Distributions

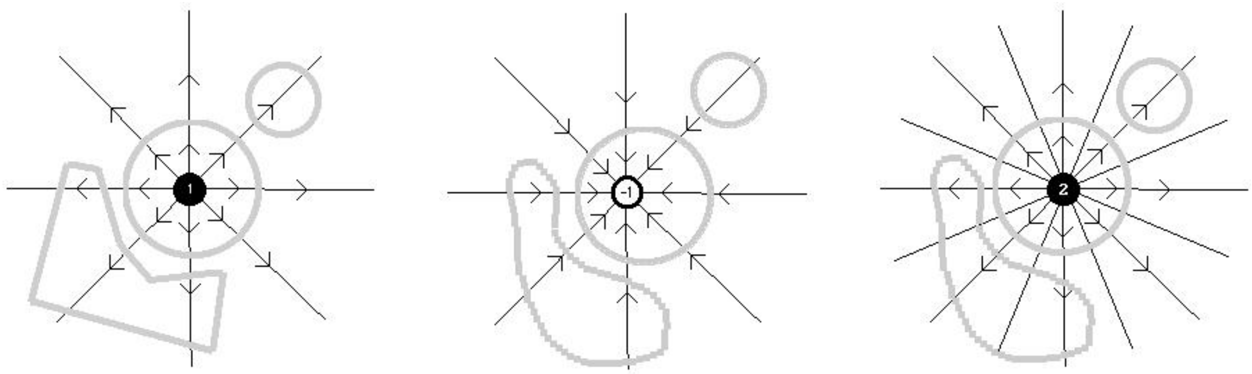
### Gauss's Law

If one draws an imaginary balloon or "bubble" anywhere in space, and if the balloon's or bubble's surface is *closed* (has no holes), then anything inside the balloon/bubble can't get out without crossing the surface. We call the net number of things that flow outwards across the surface the FLUX through the surface: FLUX = {`number out`} − {`number in`}.

When applied to electric fields and field lines, this description provides a powerful restatement of Coulomb's Law known as ***Gauss's Law***. Recall that, when using field lines to depict the electric-field pattern around a charge distribution, *one must consistently attach the same number of field lines to each positive (coming out) or negative (going in) unit of charge in the distribution*. Then—we promise—the density of field lines crossing any little virtual "window" in space is proportional to the electric field strength $E = |\boldsymbol{E}|$ there:




Let the field lines be the "flowing" things here. Suppose $n$ field lines ($n = 6$ or 8 or 12 or whatever) are attached to every $Q$'s worth of charge. An imaginary bubble-surface enclosing a positive point charge $Q$ is then pierced by $n$ field lines flowing outwards—*no matter how large the bubble!* This net number of electric-field lines flowing out through a specified surface is called the ***E***lectric flux and is denoted $\Phi_E$ (uppercase Greek Phi for "Phlux"). As long as that point charge lies inside, those radiating field lines eventually cross the bubble's surface. If twice as much *net* charge were inside the bubble, twice as many field lines would pierce the surface on their way out. If the enclosed charge were negative, the flux would be negative—the lines would be going in not out, and an *in*going flux is equivalent to an "*out*ward" flow in the opposite, negative sense. Here are examples of this:

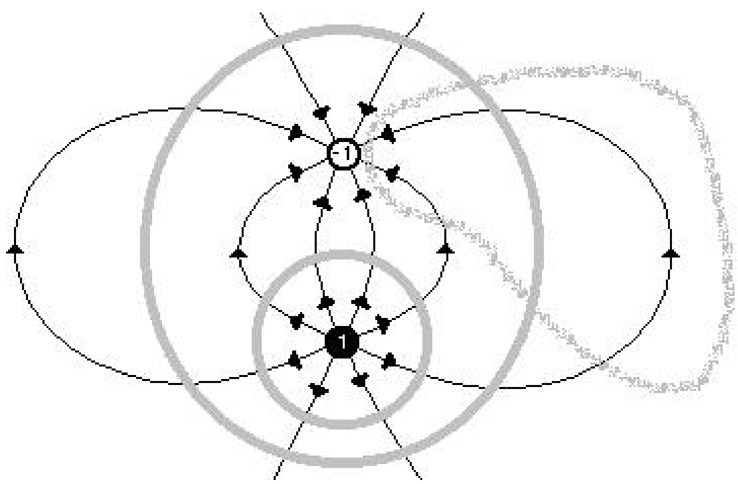



Each unit of charge here connects to 8 field lines. The single Gaussian surface enclosing each charge is pierced by +8, −8, and +16 field lines flowing outwards, respectively. (All the field lines emerging from the +2 charge are outgoing.) The other surfaces shown have a net number of ZERO field lines emerging because they enclose ZERO net charge.

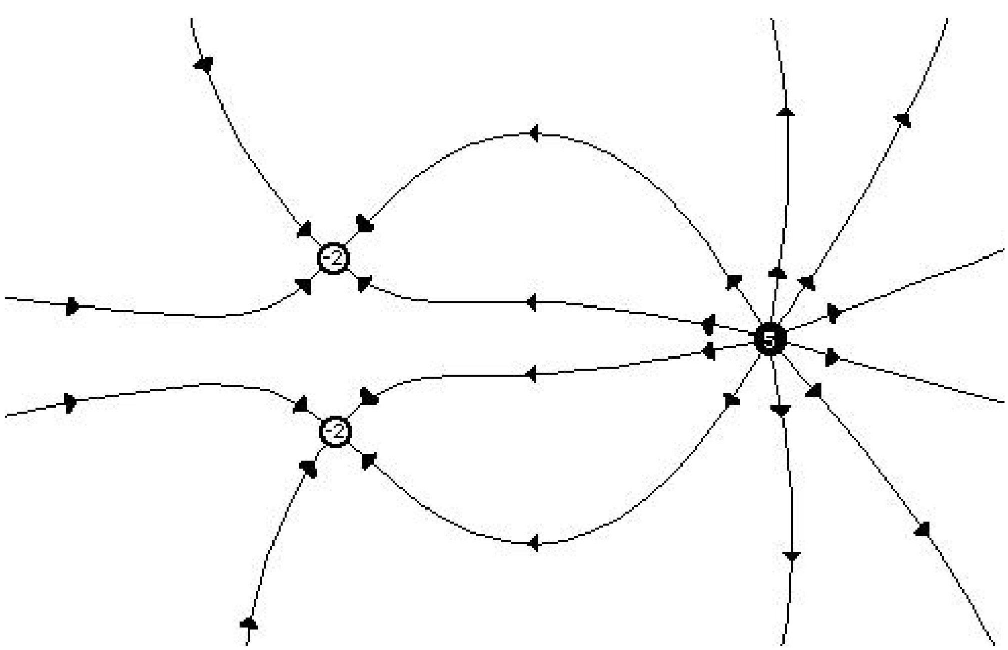
If NO NET CHARGE were enclosed in the bubble, then there would be no net number of lines coming out of or going into the surface. Each example above has two such bubbles. But this doesn't mean that there are necessarily no charges inside the bubble, only that the *net charge inside* is zero, hence the net number of "leftover" field lines is zero in this case (they're all tied up and accounted for). If some lines go out, they must eventually wind their way around and go back in.

Note that nowhere does the shape of the surface of the imaginary Gaussian balloon/bubble matter—it's completely ARBITRARY! The bubble doesn't have to be spherical or centered on any particular point. Its only constraint is that it must have no holes.

Another good example involves an electric dipole, shown below (again with 8 field lines drawn per unit charge) with one surface enclosing the $+1$ charge, a second (oval) contour enclosing both charges (net zero), and a third (irregular) enclosing no charges:




Here's one more example, with 2 field lines drawn per unit charge. The relative strengths of the charges are $-2$ units for each of the two on the left and $+5$ units for the one on the right. *See if* YOU *can verify Gauss's Law for any surface you choose to draw enclosing any (or no) charges:*



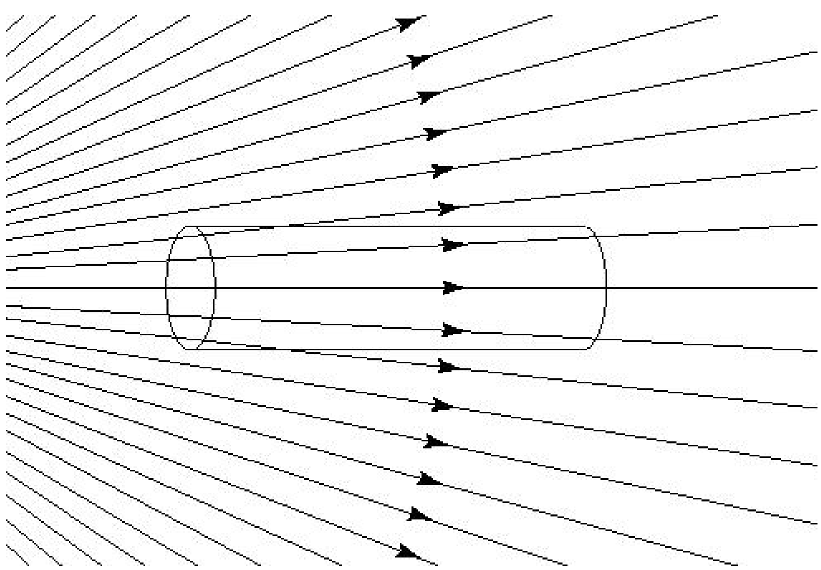
So here's Gauss's Law:

**The net flux outwards through *any* closed surface is proportional to the the total charge enclosed by the surface.**

Mathematically we represent this statement as

$$\boxed{\Phi_E \propto Q_{\text{enclosed}}}$$

We have interpreted the "electric flux" $\Phi_E$ through a surface as "net number of field lines flowing outwards" through that surface. The more precise definition of flux has us divide the whole surface into many small patches, measure the average outwards electric field over each patch, and take a kind of weighted sum of {`Electric field`} × {`patch area`} for all the patches, covering the entire surface. You can see that that's equivalent (or at least that it's a reasonable suggestion) by considering this example:



There's no net charge enclosed by this cylindrically shaped Gaussian surface, so the net outward flux of $\boldsymbol{E}$-field is ZERO. But notice that all flux enters the cylinder through the small circular face on the left where the field is strong (field lines are packed close together here)—this is negative flux that must be balanced by all the positive flux leaving the cylinder. But the positive flux leaves through the remaining area of the cylinder, which is rather larger than the circle, and this is an area of weaker field (field lines spread out). So these net fluxes are equal in magnitude, involving the same number of field lines entering or exiting the volume enclosed by the balloon. Yet one contribution has {`large field`} × {`small area`} while the other has {`small field`} × {`large area`}. Perhaps you can see that a given number of closely (widely) spaced field lines always pierce the surface through a smaller (larger) area.

For the more precise, mathematical version of Gauss's Law, refer to Appendix D of GIANCOLI/5 with these ideas in mind.

## Electric Field For Special Cases

- An observer at a distance $r$ from the center of any *spherically symmetric* charge distribution (which has three-dimensional extension) sees
$$\boldsymbol{E}(r) = \frac{1}{4\pi\epsilon_0}\,\frac{Q_<}{r^2}\,\hat{\boldsymbol{r}}$$
*as if* the total charge $Q_<$ that lies *closer* to the center than the observer's $r$ were all concentrated in a single point charge (which is zero-dimensional) situated AT THE CENTER. Any spherically symmetrically distributed charge outside of this $r$ has *zero net effect*!

- At a distance $r$ from a very long, essentially infinite line of charge (one-dimensional), with $\lambda \equiv Q/L$ charge per length ("linear charge density"), one sees
$$\boldsymbol{E}(r) = \frac{1}{2\pi\epsilon_0}\,\frac{\lambda}{r}\,\hat{\boldsymbol{r}}$$
where in this case $\hat{\boldsymbol{r}}$ points outwards from the line. The field doesn't die off as quickly with distance as the field from a lone point charge; moreover, it's independent of where you sit along the line. .. **Why?**

- A very large, essentially infinite flat plane of charge (two-dimensional), with $\sigma \equiv Q/A$ charge per area ("surface charge density"), has
$$\boldsymbol{E} = \pm\frac{1}{2\epsilon_0}\,\sigma\,\hat{\boldsymbol{z}}\,,$$
where $\pm\hat{\boldsymbol{z}}$ is the direction pointing away from the plane on either side. Most significantly, the electric field magnitude for an infinite plane of charge is *constant*, independent of BOTH the $(x, y)$ coordinates on the plane AND the height above or below it! ................. **Why?**

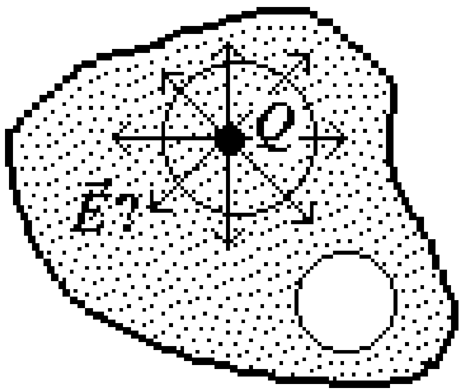
- *For a good conductor whose free charge has come to equilibrium, the electric field is precisely* $\boldsymbol{E} = 0$ *everywhere inside the conductor. Moreover, the electric field at all points on its surface must be perpendicular to the surface.* ......... **Why?** .........

## Fields and Charges in Conductors

The "statics" in Electro*statics* implies that electric charges are IN EQUILIBRIUM. The assumption that all the free charges in a perfect conductor (and even a "good" conductor) have reached equilibrium represents a severe constraint on the charges' possible configuration and $\boldsymbol{E}$-field.

IF every free charge within the body (i.e., beneath the surface) of a conductor is indeed IN EQUILIBRIUM, then it must be stable and all the forces on it must add up to ZERO. If $\boldsymbol{F}_{\text{net}} = 0$ on any one charge $q$, then it must be sitting at a spot where the electric field is ZERO: $\boldsymbol{E} = \boldsymbol{F}_{\text{net}}/q = 0$. Thus, *the electric field equals zero anywhere within the volume of a conductor in equilibrium.*

A charge sitting somewhere on the conductor's surface, however, does not have to have $\boldsymbol{F}_{\text{net}} = 0$, because we assume that ES forces cannot pull the charge out of the conductor. (This is an *assumption*; if the fields really are strong enough you certainly can pull charges out of metal. That's how electric discharges—sparks—happen. To understand the mysterious nature of how things that stick together rely on more than simple electrostatic attraction, one needs to know Quantum Mechanics, whose fuzzy matrix lies beyond this course.) *An ES force at the surface and perpendicular to the surface is acceptable*, even in an ES equilibrium situation, because an electric field at the surface yanking directly outwards on a charge sitting there is balanced by whatever "not-letting-you-pull-my-charge-off" force is provided by the conductor. On the other hand, *any component of the force parallel to the conductor's surface is forbidden* by the fact of equilibrium, since such a force would impel charges on the surface of the conductor to move around on the surface ... yet they have reached equilibrium already, so by assumption they are NOT moving around any more, so there can't be any component of the field along the surface.




**Impossible in equilibrium!**

Finally, any excess charges on a conducting object cannot be found embedded anywhere within the conducting material. For if they were to sit within the metal, we could surround every individual charge with a tiny imaginary Gaussian surface also located entirely within the metal. Field lines would have to attach to that charge (exiting or entering the Gaussian surface for a + or − charge) and yield a net flux through the surface enclosing that charge … But that would imply the existence of an electric field at least somewhere on that Gaussian surface inside the metal … which $\boldsymbol{E}$ would in turn exert forces on the conducting electrons of the metal—violating the supposed condition of equilibrium. Hence *all excess charge on a conductor in equilibrium must reside* ON ITS SURFACE.

Beware that by "surface" here we mean the OUTER surface only. This designation excludes interior surfaces for an object that might be hollow or have "bubbles" inside. So *charge cannot reside anywhere interior to the conductor*—neither on interior surfaces nor within the bulk volume of the metal ("bulk volume" refers to the material stuff itself). However, when there is some *other* charge not in contact with the conductor itself but somehow suspended in a hollow space inside the conductor, then some of the conductor's charge *can* flow onto its interior surfaces. This has the effect of canceling out this extra charge's field inside the metal and thereby maintaining $\boldsymbol{E} = 0$ in the bulk. … But we won't worry about such exotic cases and effects of "alien" charge that can't move around within the conductor itself. … Unless you're interested …

Of great practical importance is the fact that $\boldsymbol{E} = 0$ persists in a good conductor even in the face of fairly strong exterior electrostatic influences. For example, cars provide ***shielding*** for their occupants against lightning.