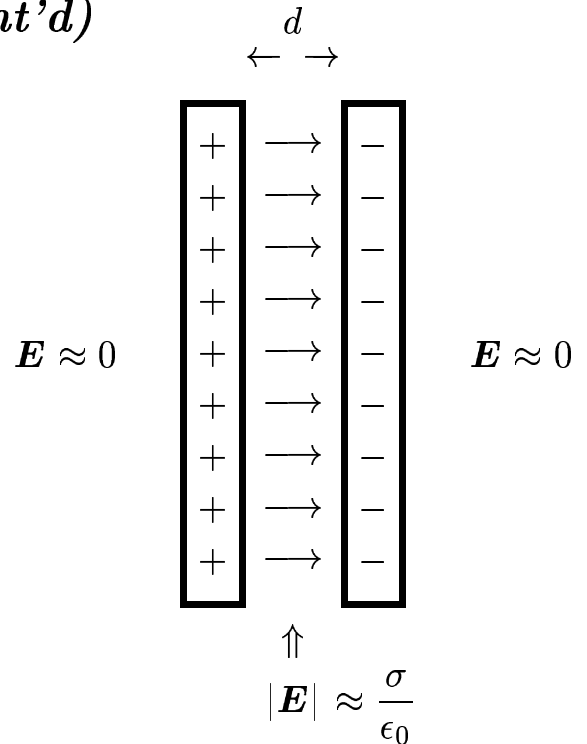
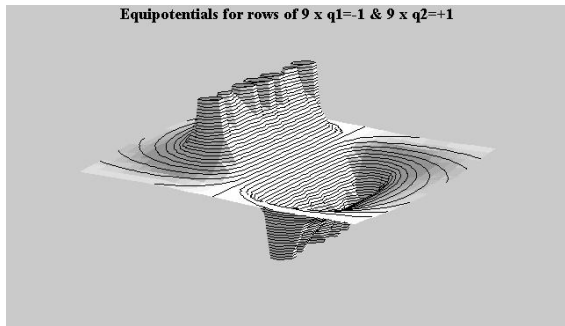


Lecture Notes #04Θ — Thu 31 Jan 2002

Potential landscapes. Capacitors & Energy Storage.

Capacitors (cont'd)



The \mathbf{E} -field nearly cancels out outside the two rows of charge but is reinforced (doubles) in between, producing $|\mathbf{E}| = 2 \times [\sigma/2\epsilon_0] = \sigma/\epsilon_0$ there.

..... → It's meant to model the case of placing together *two infinite planes of equal and opposite charge* to produce exactly zero field outside of them and a field of magnitude $|\mathbf{E}| = \sigma/\epsilon_0$ between.

“infinite” ⇒ {dimensions of charged sheets} ≫ {separation d between them}

IN REALITY they're “finite” ⇒ strength is $|\mathbf{E}| = \sigma/\epsilon_0$ only for locations not too close to the edges of the sheets. Near the edges you get small but detectable “fringing fields.”

Energy Storage in Capacitors

Imagine that we begin with two neutral conducting plates and then somehow do the work to take positive charges totalling $+Q$ from one plate and put them on the other plate; this leaves the first plate with net charge $-Q$. Such a pair of conducting plates, which may be used to hold equal-and-opposite charges, is called a parallel-plate *Capacitor*.

All this effort gets exerted by us in order to shove positive charges together while, moreover, yanking them away from a place that becomes increasingly negatively charged. This work winds up as *potential energy of the system of charges*. The PE stored in a capacitor is abbreviated U .

How much is U ? Begin with two neutral conductors. Consider how our energy expenditure adds up as we move the charges one bit at a time:

- Take the first $+$ charge from one plate and put it onto the other. That costs hardly anything since the plates were neutral.
- Put the next $+$ charge on the $+$ -charged plate. You must do some work against the repulsive force between the two $+$ charges, as well as work to pull the $+$ away from the $-$ charge left behind on the first plate. That is, you must raise the charge up the potential hill produced by the first pair of separated \pm charges. This costs.
- Put the third $+$ charge on the $+$ plate. Now you must do about twice as much work as you did in the previous step because now there's twice as much \pm charge sitting on the plates—and this charge produces about double the Coulomb force (compared to the previous step) on the charge now being transported. That is, this charge must be carried up twice the potential difference that the previous charge encountered, because now twice as much “source” charge sits on the plates already. . . .

- ... As more and more + charges are taken from one plate and placed on the other, each new bit of charge q to be transferred requires correspondingly more and more work to move it. We recognize that *the potential difference produced by the total charge $\pm Q$ already on the plates is proportional to Q* :

$$\Delta V \propto Q \quad \text{or} \quad \boxed{Q = C \Delta V}.$$

The constant of proportionality C has UNITS called “F” for “farads”; this unit is equivalent to [Coulombs]/[Volts]. The quantity C is called the **Capacitance** of this pair of conductors. All pairs of conductors—not necessarily flat or even the same shape—that might carry charges $\pm Q$ have a “capacitance” that depends on the objects’ size and geometry. Since WE must do work $W = q\Delta V$ to transfer the *next* bit of charge q between the plates, the work done ON q is also proportional to the amount of charge Q already transferred (this is what “ $\Delta V \propto Q$ ” means!):

$$W = q \Delta V = \frac{qQ}{C}$$

(Beware that the potential difference ΔV existing between the two pieces of a capacitor is often written as just ‘ V ’—e.g., “ $V = 6 \text{ V}$ ”—but don’t confuse the *variable* ‘ V ’ that we use for electric potential with the **unit** of potential, ‘V’ for Volt!!)

- All this work we do is “stored” as potential energy of the charge distribution. As we build up a net charge on the conductors, we increase the PE stored in them. If $\pm Q$ is the final amount of charge on them, $V = Q/C$ is the final potential difference between the conductors’ surfaces (and interiors, assuming they’ve attained equilibrium). However, *on the average*, the potential difference seen by all the transported charge during its migration

was neither the original value zero nor the final value V but rather the *average* value $\bar{V} = \frac{1}{2}V$. Thus, the total PE is $Q\bar{V} = \frac{1}{2}QV$. That is, *the total amount of energy stored in a capacitor is*

$$U = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$$

—3 different forms of the same thing. (CHECK THE UNITS!)

- Within a capacitor, $|\mathbf{E}| = \text{constant}$. Since $F = qE$ (in 1D), $W_{\text{by you}} = q\Delta V$, and $W_{\text{by you}} = -F\Delta x$ (in 1D), then $q\Delta V = -qE\Delta x \implies \Delta V = -E\Delta x \implies V \equiv |\Delta V| = |E|\Delta x$ — i.e., $V = Ed$ between the capacitor plates ($\Delta x = d$).
- Where did the Work $W = \frac{1}{2}QV$ go?
 - To separating the charges $+Q$ and $-Q$
 - To establishing the potential difference $V = Q/C$
 - To creating a uniform $|\mathbf{E}| = |V|/d = (Q/A)/\epsilon_0$ over the volume $A \times d \dots$ a charged capacitor is a **Box of Energy**
 - To creating an **Energy Density** $U/(Ad) = \dots = \frac{1}{2}\epsilon_0 E^2 \equiv u$
- You can show that a *parallel-plate capacitor* has capacitance

$$C_{\text{parallel plates}} = \epsilon_0 \frac{A}{d},$$

where d is the plate separation and A the plates' surface area (assuming $d \ll \sqrt{A}$).

[How? Well, ... the charge density on the plates is $\sigma = Q/A$ and the *constant* \mathbf{E} -field between them has the value $|\mathbf{E}| = \sigma/\epsilon_0 = Q/(A\epsilon_0)$, which allows us to express the charge as $Q = |\mathbf{E}|A\epsilon_0$. This \mathbf{E} -field gives rise to a potential difference of magnitude $V = |\mathbf{E}|d$ so that $C \equiv Q/V$ implies $C = (|\mathbf{E}|A\epsilon_0)/(|\mathbf{E}|d) = \epsilon_0 A/d$ (and $u = \frac{1}{2}Q^2/[CA d] = \frac{1}{2}\epsilon_0 E^2$). Q.E.D.]

Dielectrics

A “pure” capacitor is EMPTY—it consists of two oppositely charged conductors with nothing in between. Depending on the conductors’ size, shape, and separation, the potential difference between the conductors is proportional to how much charge resides on them: $V \propto Q$, which can be written equivalently as $Q/V = \text{some constant} \equiv C$. It turns out that, e.g., for parallel flat plates $C = \epsilon_0 A/d$.

But in practice the space between the two sheets of a capacitor is filled with something, like the stuff in a sandwich. What happens in that case when you place charge $\pm Q$ on the capacitor plates? In particular, how might the value of the *capacitance* C change when the capacitor is no longer filled with emptiness? Since C relates a given amount of charge on the plates ($+Q$ and $-Q$) to the change in potential in going from one plate to the other (work-per-charge V), we need to investigate how the sandwich filler affects the potential difference between the slices of bread.

Remember *polarization*? Any neutral material in the presence of external electric charges becomes *polarized* because its constituent $+$ and $-$ charges shift towards or away from the external charges. Different materials are electrically “stretchable”—polarizable—to different degrees. In this context a polarizable material is called a *dielectric* and one measure of its polarizability is called its **dielectric constant** K . This parameter ranges from $K = 1$ for a perfect vacuum (empty space) to, e.g., $K = 1.0006$ for air, $K \approx 4.3$ for glass, $K \approx 6.1$ for salt, $K \approx 80$ for H_2O , . . . , all the way to $K \rightarrow \infty$ for a perfect conductor.

Think of the electric field within the capacitor, $|\mathbf{E}| = Q/A\epsilon_0$ ($A =$ plate area). This \mathbf{E} -field urges some of the dielectric’s charges to shift, thereby segregating some $+$ from $-$ in the dielectric. Then some of

the dielectric's positive charge, an amount $+Q'$, shifts towards the capacitor plate holding $-Q$ while the opposite amount $-Q'$ shifts towards the capacitor plate holding $+Q$. In effect, the surfaces of the dielectric also become oppositely charged owing to its polarization—but with its “sheets” of charge $+Q'$ and $-Q'$ oriented OPPOSITE to the capacitor's sheets of charge $+Q$ and $-Q$. That is, if the capacitor's charges $\pm Q$ look like this $\begin{array}{c} + \\ | \\ + \\ | \\ + \\ | \\ + \\ | \\ + \\ | \\ + \end{array}$ then inside of it sits $\begin{array}{c} - \\ | \\ - \\ | \\ - \\ | \\ - \\ | \\ - \\ | \\ - \end{array}$ representing the dielectric's charges $\mp Q'$. For reasons that ultimately boil down to CONSERVATION OF ENERGY, the effective polarization charge is always less: $Q' < Q$.

The key point is that the polarized charges of the dielectric also act like two charged sheets $\pm Q'$ —like another capacitor sitting inside the “real” one and turned around. Its electric field \mathbf{E}' points in the DIRECTION OPPOSITE to the “real” capacitor's field \mathbf{E} and has a smaller magnitude ($Q' < Q \implies |\mathbf{E}'| < |\mathbf{E}|$). The net electric field between the capacitor plates—including the consequences of polarizing the dielectric—therefore comes out to be smaller in magnitude than the original field due to partial cancellation of the \mathbf{E} -fields: $|\mathbf{E}_{\text{net}}| = |\mathbf{E} - \mathbf{E}'| = (Q - Q')/(A\epsilon_0) < Q/(A\epsilon_0) = |\mathbf{E}|$. [Draw a picture of all this to convince yourself of what's going on.]

Finally, a smaller net electric field within the capacitor means that the work required to drag any charge q across from one plate to the other has decreased from the empty-dielectric case. Why? Because the force $q|\mathbf{E} - \mathbf{E}'|$ on q is less than $q|\mathbf{E}|$. Thus the work to move q the distance d between the capacitor plates is now only $W' = q|\mathbf{E} - \mathbf{E}'|d$ so the potential difference between the plates is now only $V' = W'/q = |\mathbf{E} - \mathbf{E}'|d < |\mathbf{E}|d = V$. Less potential difference $V' < V$ for the same charge Q implies a **larger capacitance**: $C' \equiv Q/V' > Q/V \equiv C$. If you define a factor $K > 1$ specifying “how much larger” the capacitance is with dielectric compared to without, you've defined the dielectric constant: $C_{\text{dielectric}} = KC_{\text{empty}}$.

There is an important physical insight to be gained here:

Insertion of a dielectric into a capacitor whose CHARGES are held fixed lowers the potential energy stored in the capacitor.

For fixed charge Q , this can be seen from $U = Q^2/2C$ and the fact that C increases. Note that $V = Q/C$ decreases, so lower U is also consistent with $U = \frac{1}{2}QV$ and $U = \frac{1}{2}CV^2$. Lower U is further consistent with the fact that the net electric field MUST decrease when the dielectric is introduced, since the energy density $u = U/Ad \propto |\mathbf{E}_{\text{net}}|^2$ decreases inside the “box”.

This entire discussion of dielectrics could have equally well made the assumption that the potential difference V between the capacitor plates was held constant—e.g., by a battery of potential V (1.5 V, 12 V, &c.)—while charge $\pm Q$ was free to flow from the plates to the battery or *vice-versa*. You’d find an identical increase in capacitance: with $V = \text{constant}$, $Q = CV$ would increase. Why? More charge would be required to maintain the same potential difference—and the same net \mathbf{E} -field—by setting up a larger “bare” field $|\mathbf{E}| = Q/A\epsilon_0$, which would get partially cancelled by the polarization field of the dielectric. But the increases in C AND Q with constant V now imply an increase in $U = \frac{1}{2}CV^2$ (or, equivalently, in $U = \frac{1}{2}QV$ and $U = Q^2/2C$). Thus,

Insertion of a dielectric into a capacitor whose POTENTIAL is held fixed raises the potential energy stored in the capacitor.

You might review the earlier discussion of work done to charge up a capacitor, and reconcile the expenditure of energy tallied there with the dielectric’s effects mentioned here. Where does the extra energy go to or come from when the dielectric is inserted into the capacitor? Who does what work on which charges?