Lecture Notes #07T — *Tue 19 Feb 2002*

Capacitors in dc Circuits

Charging a Capacitor

We have thus far investigated DC circuits that include εMFs as energy sources and resistors as energy dissipators. Electric currents—moving charges—have provided the means for borrowing energy from the εMFs and depositing that energy in the resistors. This nonequilibrium but *steady-state* situation perpetuates itself as long as the energy supply remains "exothermic."

By introducing *Capacitors* into DC circuits, we render the state of flowing charge *un*steady—i.e., the situation *does not remain constant* in time.

Imagine that a simple circuit contains just one εMF ε and one resistor R; we assume the current flows continuously at a rate given by Ohm's Law, $I = \varepsilon/R$. If a capacitor of capacitance C is inserted into the circuit, there is now a *gap* across which the current may NOT flow. We call this an ***RC* circuit**. What happens?

Why does charge flow out of the energy source—a battery, let's say—in the first place? When we complete a circuit, we connect the battery's electrodes through one or more paths, giving every charge q with excess electric potential energy PE $= q\varepsilon$ at one electrode the opportunity to lower its PE by returning to the other electrode. (Equivalently, we note that there is an electric field in the connecting wires, pointing from the $+$ to the $-$ electrode, more or less; or that opposite charges separated by the battery's PE are given the opportunity to "recombine" via a conducting path.) Charges flow at a rate $I = \varepsilon/R$ determined by how easily the intervening conductors allow charge to flow (measured by $1/R = A/\rho\ell$ of the wire/path, which is called its *conductance*). They keep flowing so long as the battery keeps pumping charges up to higher PE and the path remains CLOSED—a *closed circuit.*

Inserting the capacitor does not alter the tendency of the charge to flow but rather *interrupts* the circuit. Positive charge (conventional current) encounters one capacitor plate and accumulates on it; the same amount of positive charge leaves the other capacitor plate to return to the battery on account of CONSERVATION OF CHARGE. (Alternatively, we can say that in fact electrons pile up on the second plate as they withdraw from the first.) Current actually does continue to flow throughout the circuit—through the battery and through the resistor—although the moving charge is unable to cross the capacitor's gap. Whatever current I flows through the resistor still suffers a loss of potential of magnitude $V_R = IR$ across the resistor and steadily dissipates power $P_R = I^2R$ in it.

But the RC circuit is OPEN. Charge cannot continue to flow indefinitely as it would in a closed circuit. As charge $\pm Q$ piles onto the capacitor plates, the potential difference across the two plates $V_C = Q/C$ increases in kind. Accumulating more and more charge $\pm Q$ at an increasing potential difference V_C costs more and more and energy. This energy must be supplied by the battery, of course, and we regard it as electric potential energy "stored in" the capacitor. The battery cannot supply more energy to every charge it shoves onto the capacitor than its available energy-per-charge ε, so the potential across the capacitor approaches but cannot exceed the supplied εMF: $\boxed{V_C \to \varepsilon}$, on account CONSERVATION OF ENERGY. In the end, the work it takes to move the total accumulated amount of charge $Q = C\varepsilon$ onto the capacitor adds up to $U_C = \frac{1}{2}Q\varepsilon = \frac{1}{2}C\varepsilon^2$ (cf. LECTURE #4Θ).

Meanwhile, as charge ceases to accumulate on the plates any further, the current through the resistor drops and drops ($I \to 0$) and power ceases to be dissipated there ($P_R = I^2R \to 0$).

Now follow this chain of MUSTs:

- Charge Q MUST keep accumulating on the capacitor plates.
- The potential $V_C \propto Q$ across the capacitor MUST build up as the charge does.
- The potential V_C across the capacitor MUST not exceed ε.
- The total charge Q on the capacitor plates MUST therefore not exceed the amount $C\varepsilon$.
- As the total charges $\pm Q$ on the two plates approach magnitude $Q = C\varepsilon$, the rate $\Delta Q/\Delta t$ at which new charge is deposited on or removed from the plates MUST slow down.
- The rate of charge deposit/removal is the very definition of the current arriving at/leaving the plates, so the current $I = \Delta Q/\Delta t$ in the circuit MUST decrease.
- Every charge $q > 0$ that goes once around the circuit has potential energy $q\varepsilon$ which it MUST do something with—either give it up to the resistor or use it to perch on the high-V plate of the capacitor. Considering average energy-per-charge, the net *potential* available ε MUST go towards either V_R or V_C: $\boxed{\varepsilon = V_R + V_C = IR + Q/C}$.[*] So as Q goes up, I MUST go down.

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[*] If you know calculus, you might recognize this as a first-order differential equation for the charge $Q(t)$ in the independent variable t, since current equals $I = dQ/dt$:

$$\frac{dQ(t)}{dt}\,R + Q(t)\,\frac{1}{C} = \varepsilon\,.$$

For an initially uncharged capacitor, $Q(0) = 0$, this has solution and consequences

$$Q(t) = C\varepsilon\left[1 - e^{-t/RC}\right] \quad\Longrightarrow\quad V_C(t) = \frac{Q(t)}{C} = \varepsilon\left[1 - e^{-t/RC}\right]$$

$$\Longrightarrow\quad I(t) = \frac{dQ(t)}{dt} = \frac{\varepsilon}{R}\,e^{-t/RC} \quad\Longrightarrow\quad V_R(t) = I(t)\,R = \varepsilon\,e^{-t/RC} \underbrace{= \varepsilon - V_C(t)}_{\text{CHECK — OK!}}\,.$$

⊣⊢ Eventually the charge on the plates MUST attain its maximal value $Q = C\varepsilon$; the energy stored MUST attain its maximal value $U_C = \frac{1}{2}C\varepsilon^2$; the current I MUST drop to ZERO; and the potential drop $V_R = IR$ across the resistor and the rate of energy dissipated there, $P_R = I^2R$, MUST drop to ZERO.

To summarize: As you charge up a capacitor in an RC circuit: the potential across it rises to the final value $V_C = \varepsilon$, the charge rises to $Q = C\varepsilon$, the stored PE rises to $U_C = \frac{1}{2}C\varepsilon^2$; the current drops from an initial $I = \varepsilon/R$ to ZERO, the potential across the resistor drops from an initial $V_R = \varepsilon$ to ZERO, and the power dissipated drops from an initial rate $P_R = \varepsilon^2/R$ to ZERO.

Discharging a Capacitor

See if you can repeat the above discussion for the discharging flip side: After removing any sources of εMF, connect the plates of an initially charged capacitor C directly through a resistor R. ***What happens?*** ....

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Hint. K's Loop Rule—i.e., CONSERVATION OF ENERGY—now tells $\boxed{\mathbf{0} = I'R + Q'/C}$:

$$\frac{dQ'(t)}{dt}\,R + Q'(t)\,\frac{1}{C} = 0\,.$$

For an *initially charged* capacitor, $Q'(0) = C\varepsilon$, this has solution

$$Q'(t) = C\varepsilon\,e^{-t/RC} \quad\Longrightarrow\quad V_C{}'(t) = \frac{Q'(t)}{C} = \varepsilon\,e^{-t/RC}$$

$$\Longrightarrow\quad I'(t) = \frac{dQ'(t)}{dt} = -\frac{\varepsilon}{R}\,e^{-t/RC} \quad\Longrightarrow\quad V_R{}'(t) = I'(t)\,R = -\varepsilon\,e^{-t/RC} = \underbrace{\mathbf{0} - V_C{}'(t)}_{\text{CHECK — OK!}}\,.$$

Note that the discharging current $I' < 0$ goes in the OPPOSITE direction than in the charging case.