Physics 500: Mathematical Methods (first midterm exam)

March 11, 2007

Exam instructions This test is a three hour take-home exam which may be taken any time between now and Wed., March 14 2007 (the due date has been postponed because I have been tardy distributing the exam); the exam is due at the start of class on March 14.

Though the time limit is nominally three hours, students may take an hour or so additional time, if absolutely needed. **Note**: the three hours need not be contiguous.

If you have questions about the wording of problems on this exam, please send me email at djpriour@glue.umd.edu; I should be able to respond at least within 24 hours, though likely sooner.

Materials permitted: The course text, *Mathematical Methods for Physicists* by Arfken and Weber is permitted, as are any notes handed out or taken during class. Worked homework problems may also be consulted. However, collaboration with classmates on this exam is strictly forbidden.

1 Nonhomogeneous Second Order Differential Equations (*25 points*)

- Consider the (homogeneous) ordinary differential equation

$$my'' + ky = 0; \tag{1}$$

Show explicitly that the functions $y_1(t) = \exp(i\omega_0 t)$ and $y_2(t) = \exp(-i\omega_0 t)$ satisfy the differential equation given above *and* are linearly independent. (Hint: let's not forget about the *Wronskian*) Take ω_0 to be given by $\omega_0 \equiv \sqrt{k/m}$.

- Now, let us examine the non-homogeneous equation

$$my^{''} + ky = F\cos(\omega t), \tag{2}$$

where in general $\omega \neq \omega_0$. Exploiting the fact that we now have $y_1(t)$ and $y_2(t)$ (i.e. solutions to the *homogeneous* equation), obtain a function $y_p(t)$ which satisfies the non-homogeneous equation given above (**note**: a direct calculation is wanted; do not simply work backwards from the solution given in the next part of this problem).

- Show by direct substitution into the non-homogeneous differential equation that a valid choice for $y_p(t)$ is

$$y_p(t) = \frac{F}{m(\omega_0^2 - \omega^2)}\cos(\omega t) \tag{3}$$

2 Unitary Matrices and Eigenvalues and Eigenvectors (*25 points*)

- As we begin this part of the exam, it is useful for us to keep in mind that for a matrix $\mathbf{A}$ (matrices will be shown here in bold-faced type), a vector $\vec{v_A}$ is an *eigenvector* if we have

$$\mathbf{A}\vec{v_A} = \lambda_A \vec{v_A}, \tag{4}$$

where λ_A is the *eigenvalue* of the eigenvector $\vec{v_A}$.

- In general, a *Unitary matrix* $\mathbf{U}$ is a complex-value matrix such that

$$\mathbf{U}^\dagger \mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbf{1}, \tag{5}$$

where $\mathbf{1}$ is the identity matrix and $\mathbf{U}^\dagger$ is the *Hermitian adjoint* of $\mathbf{U}$, given by $U^\dagger_{ij} = U^*_{ji}$.

- We also recall that if all the entries of a unitary matrix $\mathbf{U_r}$ are *real*, then the unitarity condition given above reduces to

$$\mathbf{U_r}^{\mathrm{T}}\mathbf{U_r} = \mathbf{U_r}\mathbf{U_r}^{\mathrm{T}}, \text{where} \tag{6}$$

  $\mathbf{U_r}^{\mathrm{T}}$ is the matrix transpose of $\mathbf{U_r}$.

- Consider the rotation matrix

$$R = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{7}$$

- With the inner product (dot product) for planar vectors given by $\vec{v}\cdot\vec{v}' = (xx' + yy')$, show that the rotation matrix preserves the angle between two vectors; this amounts to showing explicitly that

$$\vec{v}\cdot\vec{v}' = (\mathbf{R}\vec{v})\cdot\left(\mathbf{R}\vec{v}'\right) \tag{8}$$

- Show that $\vec{v_1} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ are eigenvectors of $\mathbf{R}$. What are the corresponding eigenvalues?

- Show that an arbitary vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be written in terms of the eigenvectors as

$$\vec{v} = \frac{1}{2}(x - iy)\vec{v_1} + \frac{1}{2}(y - ix)\vec{v_2} \tag{9}$$

  Next, with $\vec{v}$ written in this manner, calculate $\mathbf{R}\vec{v}$, exploiting the fact that $\vec{v_1}$ and $\vec{v_2}$ are eigenvectors of the rotation matrix $\mathbf{R}$ (you may leave the result in terms of $\vec{v_1}$ and $\vec{v_2}$.

- Now, again representing $\vec{v}$ as a linear combination of $\vec{v_1}$ and $\vec{v_2}$, evaluate $\mathbf{R}^2\vec{v}$. Based on the results for $\mathbf{R}\vec{v}$ and $\mathbf{R}^2\vec{v}$, what expression would you anticipate for $\mathbf{R}^n\vec{v}$? (Once again, it is fine to leave the result in terms of $\vec{v_1}$ and $\vec{v_2}$ at this stage.

- Reexpress your result for $\mathbf{R}^n\vec{v}$ in terms of the components x and y of $\vec{v}$; show that the result is given by

$$\mathbf{R}^n\vec{v} = \begin{bmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{bmatrix} \tag{10}$$

3 Constructing an orthogonal basis (*25 points*)

- Consider a four dimensional Cartesian coordinate system with the general vector given by
$$\vec{v} = (v_x, v_y, v_z, v_w) \tag{11}$$
As a generalization to the three dimensional case, the inner product (i.e. the *dot product*) between vectors $\vec{v_a} = (v_{ax}, v_{ay}, v_{az}, v_{aw})$ and $\vec{v_b} = (v_{bx}, v_{by}, v_{bz}, v_{bw})$ is given by
$$\vec{v_a} \cdot \vec{v_b} = (v_{ax}v_{bx} + v_{ay}v_{by} + v_{az}v_{bz} + v_{aw}v_{bw}) \tag{12}$$

- Consider the three vectors
$$\begin{aligned} \vec{v_1} &= (1,1,1,1) \quad (13)\\ \vec{v_2} &= (1,0,1,0)\\ \vec{v_3} &= (1,2,3,4) \end{aligned}$$

- Keeping in mind that two vectors $\vec{v_a}$ and $\vec{v_b}$ are *orthogonal* if $v_a \cdot v_b = 0$, use $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ to construct a set of three mutually orthogonal vectors (**Hint**: use the Gram-Schmidt orthogonalization procedure).

4 Legendre polynomials and charge distributions with azimuthal symmetry (*25 points*)

- Consider a linear filament of length L of uniform charge density λ; to exploit the azimuthal symmetry of this line segment charge, let us orient the filament so that it coincides with the z axis (i.e. as shown in figure 1) and runs from $z = -L/2$ to $z = L/2$; the total charge is $Q = \lambda L$.

- Our task will be to calculate the electrostatic potential $V(r,\theta)$ in terms of solutions to $\nabla^2 V = 0$ appropriate to our cylindrically symmetric situation, so we will seek an expression of the form
$$V(r,\theta) = \sum_{l=0}^{\infty}(B_l r^l + A_l r^{-(l+1)})P_l(\cos\theta) = \sum_{l=0}^{\infty} A_l r^{-(l+1)} P_l(cos\theta); \tag{14}$$

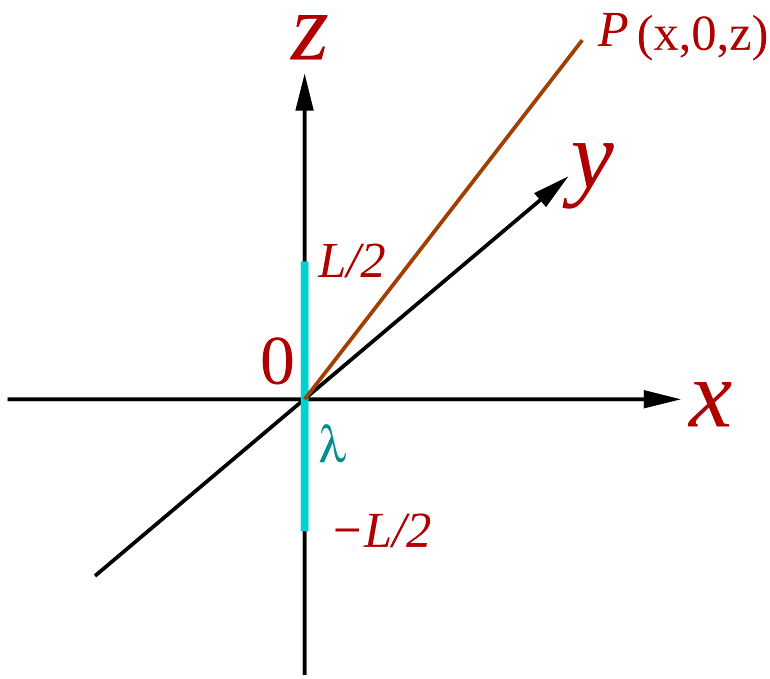

Figure 1: choosing the coordinate system to exploit azimuthal symmetry; note that $x = r\sin\theta$ and $z = r\cos\theta$.

why do we immediately know that each of the coefficients B_l vanish?

- We calculate $V(r, \theta)$ via two avenues, and we will obtain the same result in both cases, a consequence of the uniqueness property of solutions to Sturm-Liouville equations. First let us deduce the expansion coefficients indirectly. Consider a point $\mathcal{P} = (0, 0, z)$ on the z axis directly above the charged filament; note that $\theta = 0$ for this point.

- For the point $\mathcal{P}$, show by direct calculation that the electrostatic potentials is

$$\begin{aligned} V(z,0) &= K\int_{-L/2}^{L/2} \frac{\lambda dz'}{z - z'} = (KQ/L)\ln\left(\frac{z + L/2}{z - L/2}\right) \\ &= (KQ/L)[\ln(1 + L/2z) - \ln(1 - L/2z)] \end{aligned} \tag{15}$$

where it is assumed that $z > z'$ and $K \equiv 1/4\pi\epsilon_0$.

- Now, keeping in mind that

$$\ln(1+x) = x - x^2/2 + \cdots = \sum_{i=1}^{\infty} (-1)^{i+1} x^i / i, \tag{16}$$

show that $V(z,0)$ can be expanded as

$$\begin{aligned} V(z,0) &= 2KQ/L \sum_{i=1}^{\infty} (1/i)(L/2z)^i [(-1)^{i-1} - (1)^{i-1}] & (17) \\ &= 2KQ/L \sum_{j=1}^{\infty} (L/2)^{2j+1} z^{-(2j+1)}, & (18) \end{aligned}$$

so that we now have a power series in z.

- Finally, determine the cofficients A_l by equating the expansion formula $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^{-(l+1)} P_l(cos\theta)$ at $r = z$ and $\theta = 0$ to the series in powers of z which you obtained earlier; show that

$$V(r,\theta) = \sum_{j=0}^{\infty} \frac{KQ}{r} (L/2r)^{2j} P_{2j}(cos\theta) \tag{19}$$

Hint: keep in mind that $P_l(1) = 1$.

- Let's carry out the same calculation, only in a more direct manner this time. The task now is to calculate V off of the z axis. We will lose no generality by concentrating on the point $\mathcal{P} = (x,0,z) = (r\sin\theta, 0, r\cos\theta)$; show that the electrostatic potential at $\mathcal{P}$ is given by

$$V(x,0,z) = V(r,\theta) = K \int_{-L/2}^{L/2} \frac{\lambda dz'}{\sqrt{x^2 + z^2 + (z')^2 - 2zz'}}; \tag{20}$$

finally, show that this expression leads to a formula for $V(r,\theta)$ identical to that given in Equation 19. To work toward this goal, it will be useful to express x and z in terms of spherical polar coordinates as needed. Recall that the generating function for the legendre polynomials can be expanded as

$$(1 - 2tx + t^2)^{-1/2} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{21}$$

You should obtain an integral of a series, and it may be helpful to note that you are free to exchange the order of the integration and the summation of the series; integrating term by term should lead to the desired result.