

## Appendix - Propagation of Errors ( Calculus Method )

### PROPAGATION OF ERRORS

When taking measurements in the laboratory, there is an uncertainty associated with each measurement. All of the measurements are then combined in some way to give a final experimental value. Since there is an uncertainty in the individual measurements, there has to be an uncertainty in the final answer. The correct way to find this associated uncertainty of the final answer is through the method of propagation of errors. In this method, which is described below, you will see how to take into account these individual measurement errors and how they are propagated through to the final answer.

This technique requires the use and knowledge of partial derivatives. Since you may or may not know how to take partial derivatives, we will teach you the method, but not the theory. The theory will be taught in Calculus III and you do not need to be exposed to it to understand the techniques of taking partial derivatives. The method of taking partial derivatives is very similar to taking the derivative of a function with a single variable.

The best way to learn how to do partial derivatives is by example. The important thing to notice here is that when taking the partial derivative of a function of several variables with respect to one of those variables, you treat all of the other variables as constants. This should hopefully become clear in the following example.

*Example 1* - Let's look at the function

$$f(x,y,z) = 2xy^2z^3 + \{x^3/z^2\} = 2xy^2z^3 + x^3z^{-2} \quad (14.1)$$

As you can see, the function depends on three variables. This means that there are three partial derivatives that you can take of this function  $f$ . The proper way to write the partial derivative of  $f$  with respect to the variable  $x$  is

$$\partial f(x,y,z)/\partial x = \partial f/\partial x \quad (14.2)$$

Now, let us take the partial derivatives of this function.

$$\begin{aligned} \partial f/\partial x &= 2y^2z^3 + 3x^2/z^2 \\ \partial f/\partial y &= 4xyz^3 + 0 \\ \partial f/\partial z &= 6xy^2z^2 - 2x^3/z^3 \end{aligned} \quad (14.3)$$

Now that you understands how to take partial derivatives, we will now learn how to propagate errors. This appears to be a complicated numeric technique, but don't let it intimidate you because it is very simple once you understand it. The best way to understand it is to see it applied to an example. If after seeing this example, you are still confused, ask your instructor for help.

If  $g = f(x_1, x_2, \dots, x_n)$  then the final error in this value of  $g$  is  $\Delta g$ . It can be shown that,

$$\Delta g = [(\partial f / \partial x_1 * \Delta x_1)^2 + (\partial f / \partial x_2 * \Delta x_2)^2 + \dots + (\partial f / \partial x_n * \Delta x_n)^2]^{1/2} \quad (14.4)$$

In this equation  $(\partial f / \partial x_i) \Delta x_i$  is the uncertainty of  $f$  due to the measurement of  $x_i$ . Also, the correct way to report the final value when reporting a final value for an experiment is to write  $g \pm \Delta g$  with the correct units. This equation is a very important and you should make sure that you remember it and understand how to use it. In this equation we have assumed that the measurements of  $x_1, x_2, \dots, x_n$  are all uncorrelated<sup>1</sup>. In this lab we will always assume that the errors are uncorrelated, so Eq.(14.4) will always be the way to find the error of your final answer.

The way to find what values to use for  $\Delta x_1, \Delta x_2, \dots$  may seem a bit confusing. Actually this is very simple: you just have to think about the way that the measurement of  $x_n$  was taken.

In this lab, you should have one of the cases listed below.

The value of  $\Delta x_n$

- is the least count of the measuring instrument divided by two if only one measurement is taken and it is taken by using the instrument in its most accurate way.
- can be whatever you think the estimated uncertainty in the measurement was.
- is the value of the standard deviation of several (approximately 10 or more) measurements for one value, where in this case the actual value of  $\Delta x_n$  would be the average of these values.
- can be determined from your graph.

In doing an experiment, you should choose which of the above ways is the best way to find the error in a measurement. The method of doing the analysis determines which way the error should be found.

*Example 2* - Take the function  $V = \pi s^2 h / 4$ , the function for the volume of a cylinder where  $s$  is the diameter of the cylinder and  $h$  is its height. In this equation the values of 4 and  $\pi$  are constant, so  $V$  can be written as  $V(s, h) = \pi s^2 h / 4$ . Assume that you are told to measure the height value once and the diameter value ten times. Now let us do the propagation of error of this equation by using the method described above:

$$\Delta V = \sqrt{[(\partial V / \partial s * \Delta s)^2 + (\partial V / \partial h * \Delta h)^2]} \quad (14.5)$$

where the partial derivatives are given by

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<sup>1</sup> All this means is that the measurements do not depend on one another. For example, if you were measuring the length and width of a sheet of paper, whatever value you got for the length would have no effect on and make no difference to the value you would measure for the width.

$$\partial V/\partial s = \pi sh/2, \text{ and } \partial V/\partial h = \pi s^2/4, \quad (14.6)$$

and

$$\Delta s = \sigma_{s,d} \text{ and } \Delta h = L.C./2, \quad (14.7)$$

where the value of  $\Delta s$  is obtained from the sample standard deviation of the ten diameter values (as in the second lab) and the value of  $\Delta h$  is the least count (L.C.) of the measuring instrument divided by two. The following values were measured in the lab:

$$s = s_{ave} = 2.058 \text{ cm, where } \Delta s = \sigma_{s,d} = 0.00043 \text{ cm} \quad (14.8)$$

and

$$h = 1.443 \text{ cm, where } \Delta h = L.C.(\text{micrometer})/2 = 0.001\text{cm}/2 = 0.0005 \text{ cm} \quad (14.9)$$

Substituting these numbers in the equation for the volume, we get

$$V = \pi/4(2.058\text{cm})^2(1.443 \text{ cm}) = 4.800063 \text{ cm}^3 \quad (14.10)$$

Now we must find the error associated with this value. The error can be found by using Eq. 14.5 which gives

$$\Delta V = [ (0.0020059 \text{ cm}^3)^2 + (0.0016632 \text{ cm}^3)^2 ]^{1/2} = 0.00261 \text{ cm}^3, \quad (14.11)$$

where

$$\partial V/\partial s * \Delta s = \pi sh/2 * \Delta s = \pi/2 * (2.058\text{cm})(1.443\text{cm})(0.00043\text{cm}) = 0.0020059 \text{ cm}^3 \quad (14.12)$$

and

$$\partial V/\partial h * \Delta h = \pi s^2/4 * \Delta h = \pi/4 * (2.058\text{cm})^2(0.0005 \text{ cm}) = 0.0016632 \text{ cm}^3 \quad (14.13)$$

Now we have found the experimental value of  $V$  and its error  $\Delta V$ . All that is left for us to do is to report this value in an acceptable form. There are two widely accepted forms:

$$V = 4.8001 \text{ cm}^3 \pm 0.0026 \text{ cm}^3$$

OR

$$V = (4.8001 \pm 0.0026)\text{cm}^3. \quad (14.14)$$

This is the way to do error propagation properly and the correct way to report it. When you are writing up your lab report, you should always write out Eq. 4 with the proper variables. Also, you should always write out your partial derivatives as in Eq. 6. This is so your TA can give you partial credit in case you make a mistake. Another reason to write out the partial derivatives multiplied by the appropriate error is that it will let you see what term made the biggest

contribution to your final error. This Appendix feeds information into the Appendix- Errors and Uncertainties, but does not supercede it

### **Bibliography**

Bevington, Philip R. *Data Reduction and Error Analysis for the Physical Sciences*, McGraw--Hill, New York 1969.

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