Rate of Occurrence

In many experiments, though we measure the number of occurrence of an event, but what we really want to know is rate of occurrence of the event \( R \).

Since we are measuring the number of occurrence over a predetermined fix time period \( T \), so \( R = \frac{\nu}{T} \).

Propagation of error:

\[
R = \frac{\nu}{T} \quad \Rightarrow \quad \delta R = \frac{\delta \nu}{T}
\]

\[∴ \text{ Fractional uncertainty :} \]
\[
\frac{\delta R}{R} = \frac{\delta \nu}{\nu}
\]

If you do the measurement many times and take average,

\[\nu \sim \mu \quad \text{and} \quad \delta \nu \sim \sqrt{\mu} \]

\[∴ \quad \frac{\delta R}{R} \sim \sqrt{\mu} \approx \frac{1}{\mu} \]
To Increase Precision in the Measurement of the Rate of Occurrence

Since \( \frac{\delta R}{R} \sim \frac{1}{\sqrt{\mu}} \), the fractional error in the rate of occurrence will be reduced if the \( \mu \) (mean) of our model Poisson distribution is increased. This can be done if we realize that \( \mu \propto T \). In other words, we have to increase the time period of measurement:

\[
\frac{\delta R}{R} \sim \frac{1}{\sqrt{\mu}} \quad \text{and} \quad \mu \propto T \quad \Rightarrow \quad \frac{\delta R}{R} \propto \frac{1}{\sqrt{T}}
\]
Adding/Subtracting Independent Rates

Sample problems:

1. If we measure the radioactive rate of source A \( (R_A) \) and that of source B \( (R_B) \) separately, can we predict the total radioactive rate \( (R_{Tot}) \) when we put two sources together?

2. In an experiment we try to measure the radioactive rate of a source \( (R_{sce}) \) but it is difficult to do so because there is radioactive background. So we measure the radioactive rate of the source together with the background \( (R_{Tot}) \), then remove the source and measure the radioactive rate of the background alone \( (R_{bgd}) \). Can be obtain \( R_{sce} \) from \( R_{Tot} \) and \( R_{bgd} \)?
Adding/Subtracting Independent Rates

1. In general, we cannot add the number of counts ($\nu_A$ and $\nu_B$) because the two measurements may have different time period of measurement. However, we can easily add rates together:

   $$ R_{\text{tot}} = R_A + R_B $$

2. We need to be careful in estimating the uncertainty. To estimate the uncertainty in any of the above three terms, we need to know the uncertainties of the other two terms first (this can be known from the number of counts $\nu$ and the time of measurement $T$) and then use error propagation. For example,

   $$ \delta R_{\text{tot}} = \delta R_A + \delta R_B = \frac{\sqrt{\nu_A}}{T_A} + \frac{\sqrt{\nu_B}}{T_B} $$
Adding/Subtracting Independent Rates

Example (p.254 of text)

A student decides to monitor the activity of a radioactive source by placing it in a liquid scintillation detector. In the course of 10 minutes, the detector registers 2540 total counts. To allow for the possibility of unwanted background counts, she removes the source and notes that in 3 minutes, the detector registers a further 95 counts. What is the source activity with uncertainty?

Given $\nu_{\text{Tot}} = 2540$, $\therefore \delta \nu_{\text{Tot}} = \sqrt{2540} \sim 50$

$\therefore R_{\text{Tot}} = \frac{2540 \pm 50}{10} = (254 \pm 5) \text{ counts / min}$

Given $\nu_{\text{bgd}} = 95$, $\therefore \delta \nu_{\text{bgd}} = \sqrt{95} \sim 10$

$\therefore R_{\text{bgd}} = \frac{95 \pm 10}{3} = (32 \pm 3) \text{ counts / min}$

$\therefore$ The activity of the sourced is

$R_{\text{sce}} = R_{\text{Tot}} - R_{\text{bgd}} = 254 - 32 = 222 \text{ counts /min, with an uncertainty}$

$\delta R_{\text{sce}} = \delta R_{\text{Tot}} + \delta R_{\text{bgd}} = 5 + 3 = 8 \text{ counts /min}$

$\therefore$ The activity of the sourced is $(222 \pm 8) \text{ counts / min}$

(To be more accurate, $\delta R_{\text{sce}}$ should equal to $\sqrt{5^2 + 3^2} = 6 \text{ counts/min, but we are not there yet!}$)
Poisson Distribution in time scale

Sometimes process can occur very fast and many events can occur within the time of interest. In this case, the process itself is a “fast machine” and we do not need to rely on the population replica to do the measurement.

Let \( T = \) time of interest which is fix for our experiment. The random variable is \( \nu = \) number of events in time interval \([0,T]\).

Since we are using the time scale, it is more convenient to define the relaxation time \( \tau = \) Average time interval between two events. \( \mu \) can be replaced by \( \tau : \)

\[ \mu = \text{mean of number of events} = \frac{T}{\tau} \]

We know \( \nu \) will follow Poisson distribution

\[ P_\mu(\nu) = e^{-\mu} \frac{\mu^\nu}{\nu!} \]

or in terms of \( \tau : \)

\[ P_{\tau}(\nu) = e^{\frac{T}{\tau}} \left( \frac{T}{\tau} \right)^\nu \frac{1}{\nu!} \]
Example

An electron moving in a metal makes $10^{14}$ collisions with the atoms in the lattice in one second, so $\mu = 10^{14}$.

$\tau = 1\text{s}/ 10^{14} = 10^{-14}$ is also known as the “relaxation time”.

The number of collisions made in one second follows Poisson distribution

$$P_{1/\tau}(\nu) = \frac{e^{-\frac{1}{10^{-14}}}}{\nu!} \left( \frac{1}{10^{-14}} \right)^\nu = \frac{e^{-10^{14}}}{\nu!} 10^{14\nu}$$

The number of collisions made in two seconds also follows Poisson distribution

$$P_{2/\tau}(\nu) = \frac{e^{-\frac{2}{10^{-14}}}}{\nu!} \left( \frac{2}{10^{-14}} \right)^\nu = \frac{e^{-2\times10^{14}}}{\nu!} (2\times10^{14})^\nu$$
Topics

Part 1 – Single measurement

1. Basic stuff (Chapter 1 and 2)
2. Propagation of uncertainties (Chapter 3)

Part 2 – Multiple measurements as independent results

1. Mean and standard deviation (Chapter 4)
2. Basic on probability distribution function (not in text explicitly)
3. The Binomial distribution (Chapter 10)
4. The Poisson distribution (Chapter 11)
5. Normal distribution (first half of Chapter 5)
6. \( \chi^2 \) test – how well does the data fit the distribution model? (Chapter 12)

Part 3 – Multiple measurements as one sample

1. Central limit theorem (not in text explicitly)
2. Normal distribution (second half of Chapter 5)
3. Propagation of error (Chapter 3)
4. Rejection of data (Chapter 6)
5. Merging two sets of data together (Chapter 7)

Part 4 - Dependent variables

1. Curve fitting (Chapter 8)
2. Covariance and correlation (Chapter 9)
Normal Distribution (aka Gaussian Distribution)

$$G_{X,\sigma}(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}}$$
Normalization of Normal Distribution

Given \( P(x) = G_{x,\sigma}(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}} \)

\[
\int_{-\infty}^{\infty} P(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}} dx
\]

Substitution \( r = \frac{(x - X)}{\sqrt{2}\sigma} \Rightarrow dr = \frac{dx}{\sqrt{2}\sigma} \Rightarrow dx = \sqrt{2}\sigma dr \)

Also note that \( r = \pm \infty \) when \( x = \pm \infty \)

\[
\therefore \int_{-\infty}^{\infty} P(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-r^2} \cdot \sqrt{2}\sigma dr
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} dr
\]

But how to do the integral \( \int_{-\infty}^{\infty} e^{-r^2} dr \) ?
The indefinite integral \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \) cannot be done analytically. But the definite integral \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \) can be calculated analytically without any approximation.

Since the indefinite integral \( \int e^{-x^2} \, dx \) cannot be done analytically, we package it as a function called error function:

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-x^2} \, dx
\]
erf (z)

1. All values of erf(z) are calculated numerically, except when \( z = \pm \infty \).

2. erf(z) is an odd function:
   \[ \text{erf}(z) = -\text{erf}(-z) \]

3. erf(0) = 0 (has to be the case for odd function)

4. erf(\( \infty \)) = 1 and erf(-\( \infty \)) = -1

5. Complementary error function erfc is defined as:
   \[ \text{erf}(z) + \text{erfc}(z) = 1 \]

6. \[
   \int_{0}^{z} G_{X, \sigma}(x) \, dx = \int_{0}^{z} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}} \, dx = \frac{1}{2} \text{erf}(z)
   \]
   More generally:
   \[
   \int_{a}^{b} G_{X, \sigma}(x) \, dx = \int_{a}^{b} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}} \, dx = \frac{1}{2} [\text{erf}(b) - \text{erf}(a)]
   \]
\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx \]

We can do this definite integral by three methods. The first method is by the error function:

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = \int_{-\infty}^{0} e^{-x^2} \, dx + \int_{0}^{\infty} e^{-x^2} \, dx = -\int_{0}^{-\infty} e^{-x^2} \, dx + \int_{0}^{\infty} e^{-x^2} \, dx
\]

\[
= -\frac{\sqrt{\pi}}{2} \text{erf}(-\infty) + \frac{\sqrt{\pi}}{2} \text{erf}(\infty)
\]

\[
= -\frac{\sqrt{\pi}}{2} (-1) + \frac{\sqrt{\pi}}{2} (1)
\]

\[
= \sqrt{\pi}
\]

\[
\therefore \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}
\]

Of course, we are not really doing anything here because we have assumed \( \text{erf}(\infty) = 1 \) and \( \text{erf}(-\infty) = -1 \), which requires proof at the end.
\[
\int_{-\infty}^{\infty} e^{-r^2} \, dr
\]

The second method is a beautiful trick, but this is a proof!

Let \( I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \)

So \( I^2 = \int_{-\infty}^{\infty} e^{-x^2} \, dx \int_{-\infty}^{\infty} e^{-y^2} \, dy = \iint_{\text{whole } x-y \text{ plane}} e^{-(x^2+y^2)} \, dx \, dy \)

Change from Cartesian coordinates to polar coordinates \( x^2 + y^2 = r^2 \) and \( dx \, dy = r \, dr \, d\theta \)

\[ I^2 = \iint_{\text{whole } x-y \text{ plane}} e^{(x^2+y^2)} \, dx \, dy = \iint_{\text{whole } x-y \text{ plane}} e^{-r^2} \, r \, dr \, d\theta \]

\[ = \int_{0}^{\infty} e^{-r^2} \, r \, dr \int_{0}^{2\pi} d\theta \]

Substitution \( r^2 = u \Rightarrow 2r \, dr = du \Rightarrow r \, dr = \frac{du}{2} \)

\[ \therefore I^2 = \frac{1}{2} \int_{0}^{\infty} e^{-u} \, du \int_{0}^{2\pi} d\theta = \frac{1}{2} \left[-e^{-u}\right]_{0}^{\infty} \cdot 2\pi = \pi \]

\[ \therefore I = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \]
\[ \int_{-\infty}^{\infty} e^{-r^2} \, dr \]

The third method is not a real proof either, but it is a powerful tool to do integration of the form

\[ \int_{0}^{\infty} x^m e^{-x^n} \, dx \]

This involves the use of Gamma function \( \Gamma(z) \).
1. Definition of $\Gamma(n)$:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

2. For positive $z$: $\Gamma(z+1)=z\Gamma(z)$, except or non-positive integers.

3. $\Gamma(1)=1$

4. $\Gamma(n+1)=n!$ for all positive integers $n$.

5. $\Gamma(n)$ is divergent at $n=0$ and all negative integers.

6. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$
\[
\int_{-\infty}^{\infty} e^{-r^2} \, dr
\]

Now we can do this integration with Gamma function:

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = 2\int_{0}^{\infty} e^{-x^2} \, dx
\]

Let \( x^2 = z \) \( \Rightarrow \) \( 2x \, dx = dz \) \( \Rightarrow \) \( dx = \frac{1}{2} x^{-1} \, dz \) \( \Rightarrow \) \( dx = \frac{1}{2} z^{-\frac{1}{2}} \, dz \)

\[
\therefore \int_{0}^{\infty} e^{-x^2} \, dx = \frac{1}{2} \int_{0}^{\infty} z^{-\frac{1}{2}} e^{-z} \, dz
\]

\[
= \frac{1}{2} \Gamma\left( \frac{1}{2} \right)
\]

\[
= \frac{\sqrt{\pi}}{2}
\]

\[
\therefore \int_{-\infty}^{\infty} e^{-x^2} \, dx = 2\int_{0}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}
\]
Normalization of Normal Distribution

Given \( P(x) = G_{x, \sigma}(x) = \frac{1}{\sigma \sqrt{2 \pi}} e^{\frac{(x-X)^2}{2\sigma^2}} \)

\[
\int_{-\infty}^{\infty} P(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2 \pi}} e^{\frac{(x-X)^2}{2\sigma^2}} dx
\]

Substitution \( r = \frac{(x - X)}{\sqrt{2\sigma}} \Rightarrow dr = \frac{dx}{\sqrt{2\sigma}} \Rightarrow dx = \sqrt{2\sigma}dr \)

Also note that \( r = \pm \infty \) when \( x = \pm \infty \)

\[
\therefore \int_{-\infty}^{\infty} P(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2 \pi}} e^{-r^2} \cdot \sqrt{2\sigma}dr
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} dr
\]

But how to do the integral \( \int_{-\infty}^{\infty} e^{-r^2} dr \)?
Normalization of Normal Distribution

Now we can easily show the normal distribution function is normalized:

\[ \int_{-\infty}^{\infty} G_{X,\sigma}(x)dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} dr \]

We have already shown that \( \int_{-\infty}^{\infty} e^{-r^2} dr = \sqrt{\pi} \)

\[ \therefore \int_{-\infty}^{\infty} G_{X,\sigma}(x)dx = 1 \]
Mean of Normal Distribution

\[ \bar{x} = \int_{-\infty}^{\infty} xP(x)dx = \int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}} dx \]

Substitution \( r = \frac{x - X}{\sqrt{2\sigma}} \Rightarrow x = \sqrt{2\sigma} r + X \Rightarrow dx = \sqrt{2}\sigma dr \)

Also note that \( r = \pm\infty \) when \( x = \pm\infty \)

\[ \therefore \int_{-\infty}^{\infty} xP(x)dx = \int_{-\infty}^{\infty} \frac{\sqrt{2\sigma} r + X}{\sigma\sqrt{2\pi}} e^{-r^2} \cdot \sqrt{2}\sigma dr \]

\[ = \int_{-\infty}^{\infty} \frac{\sqrt{2\sigma} r + X}{\sigma\sqrt{2\pi}} e^{-r^2} \cdot \sqrt{2}\sigma dr \]

\[ = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} re^{-r^2} dr + \int_{-\infty}^{\infty} \frac{X}{\sqrt{\pi}} e^{-r^2} dr \]

\[ = 0 + \frac{X}{\sqrt{\pi}} \cdot \sqrt{\pi} \]

\[ = X \]

\[ \therefore \bar{x} = X \]
Standard Deviation of Normal Distribution

\[
\text{Variance} = \int_{-\infty}^{\infty} (x - \bar{x})^2 P(x) \, dx = \int_{-\infty}^{\infty} \frac{(x - X)^2}{\sigma \sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}} \, dx
\]

Substitution \( r = \frac{(x - X)}{\sqrt{2}\sigma} \Rightarrow x = \sqrt{2} \sigma r + X \Rightarrow dx = \sqrt{2}\sigma \, dr \)

Also note that \( r = \pm\infty \) when \( x = \pm\infty \)

\[
\therefore \quad \text{Variance} = \int_{-\infty}^{\infty} \frac{2\sigma^2 r^2}{\sigma \sqrt{2\pi}} e^{-r^2} \cdot \sqrt{2}\sigma \, dr
\]

\[
= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} r^2 e^{-r^2} \, dr
\]

\[
= 2 \times \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} r^2 e^{-r^2} \, dr
\]

Let \( r^2 = u \Rightarrow 2rdr = du \Rightarrow dr = \frac{1}{2} \cdot u^{-\frac{1}{2}} \, du \)

\[
\therefore \quad \text{Variance} = \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} u^{-\frac{1}{2}} e^{-u} \cdot \frac{1}{2} \cdot u^{\frac{1}{2}} \, du
\]

\[
= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{2} \cdot 1 \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \, du
\]

\[
= \frac{\sigma^2}{\sqrt{\pi}} \cdot \sqrt{\pi}
\]

\[
= \sigma^2
\]

\[
\therefore \text{Standard deviation} = \sqrt{\text{Variance}} = \sigma
\]