Blackbody radiation (Text 2.2)

How Raleigh and Jeans model the problem:

2. Next step is to calculate how many possible independent standing waves are there *per unit frequency* (ν) per unit volume (of cavity). It is easier to calculate the number of possible independent standing waves (states) per unit k first, since k is related to ν.

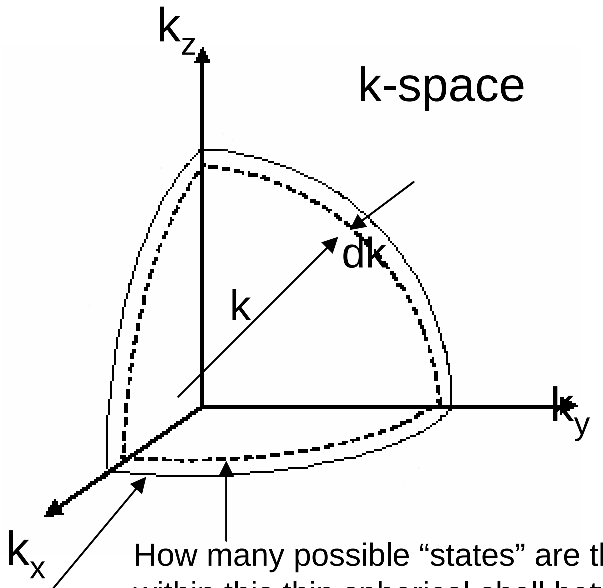
Note that the states are packed together close together uniformly in the k-space, because L is large.

With $\Delta n=1$, Volume per state in k-space

$$=\left(\frac{\pi}{L_x}\right)\left(\frac{\pi}{L_y}\right)\left(\frac{\pi}{L_z}\right)=\frac{\pi^3}{L_xL_yL_z}=\frac{\pi^3}{V}\quad (V = \text{volume of cavity})$$

Therefore, the total number of possible standing waves within the spherical shell

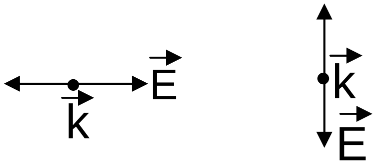
$$=\underset{\substack{\uparrow\\ \text{2 possible polarizations}}}{2}\times\underset{\substack{\uparrow\\ \text{1/8 of a complete sphere}}}{\frac{1}{8}}\times\frac{\text{Volume of spherical shell}}{\text{Volume per state}}$$



How many possible "states" are there within this thin spherical shell between radii k and k+dk?

Each "state" represent a possible standing wave defined by wave vector (k_x, k_y, k_z) in the cavity.

There are two possible polarizations for a transverse standing wave:



Blackbody radiation (Text 2.2)

How Raleigh and Jeans model the problem:

2. (Con't) The total number of possible standing waves within the spherical shell

$$= 2\times\frac{1}{8}\times\frac{\text{Volume of spherical shell}}{\text{Volume per state}}$$

$$= 2\times\frac{1}{8}\times 4\pi\,k^2dk \Big/ \frac{\pi^3}{V}$$

$$= \frac{\pi\,k^2Vdk}{\pi^3}$$

But k is related to ν:

$\nu\lambda = c \Rightarrow 2\pi\nu/c = 2\pi/\lambda \Rightarrow k = 2\pi\nu/c$

Therefore, the total number of standing waves between ν and $\nu+d\nu$ is:

$$\frac{\pi\,k^2Vdk}{\pi^3} = \frac{\pi\left(\frac{2\pi\nu}{c}\right)^2 V\,\frac{2\pi\,d\nu}{c}}{\pi^3} = \frac{8\pi\nu^2 d\nu}{c^3}V$$

Definition:

Density of states G(E) is the number of states per unit energy E per unit volume V of sample,

$\therefore$ Number of states between energy E and E+dE

$= G(E)dE \times V$

In present case,

Density of standing waves $G(\nu)$ is the number of possible standing waves per unit frequency ν per unit volume V of cavity,

$\therefore$ Number of possible standing waves between frequencies ν and $\nu + d\nu$

$= G(\nu)d\,\nu \times V$

$$\text{Equal} \Rightarrow \frac{8\pi\nu^2 d\nu}{c^3}V = G(\nu)d\nu V$$

$$\Rightarrow G(\nu)d\nu = \frac{8\pi\nu^2 d\nu}{c^3} \qquad \text{(Eq. (2.1) of text)}$$

$$\text{or} \qquad G(\nu) = \frac{8\pi\nu^2}{c^3}$$

Blackbody radiation (Text 2.2)

How Rayleigh and Jeans model the problem:

3. Density of wave function allows us to calculate the total of any physical quantity (say, $f(\nu)$) that is a function of frequency:

Total f for the whole system = $V\int_0^\infty f(\nu)G(\nu)d\nu$

An example is energy $E(\nu)$, but how does E depends on ν?

Rayleigh assumed the classical law of equipartition energy. He said, "one dimensional waves always have 2 degrees of freedom, one for potential energy (x) and the other for kinetic energy (v). In case of electromagnetic wave, these two degree of freedom are derived from electric field and magnetic field."

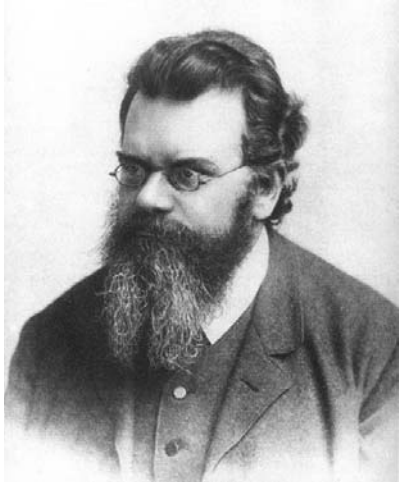
This is THE fatal assumption!

Classically, each degree of freedom has an energy of $k_BT/2$. So the energy for each standing wave is k_BT:

$$E(\nu) = k_BT$$

k_B is the Boltzman constant:

$k_B = 1.381\times10^{-23}$ J/K or 0.08617 meV/K



Boltzman

Blackbody radiation (Not in text)

If you understand the previous slide, you should be able to answer the following sample GRE problem:

A three-dimensional harmonic oscillator is in thermal equilibrium with a temperature reservoir at temperature *T*. The average total energy of the oscillator is
(A) 1/2*kT*
(B) *kT*
(C) 3/2*kT*
(D) 3*kT*
(E) 6*kT*

Blackbody radiation (Text 2.2)

How Raleigh and Jeans model the problem:

4. Final touch.

If we define the spectral density as the energy radiated per unit frequency per unit volume.

$$\therefore \quad u(\nu)d\nu V = G(\nu)E(\nu)d\nu\ V$$

$$\Rightarrow u(\nu) = G(\nu)E(\nu)$$

$$\Rightarrow u(\nu) = \frac{8\pi\nu^2 k_B T}{c^3}$$ ⟵Rayleigh-Jeans Formula

Note how this follows Wien's Law:

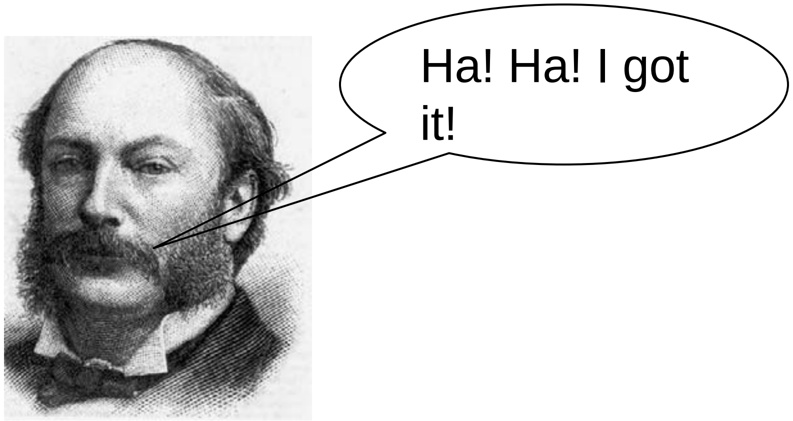
$$u(\nu)\,|\,d\nu\,| = u(\lambda)\,|\,d\lambda\,| \Rightarrow u(\lambda) = u(\nu)\left|\frac{d\nu}{d\lambda}\right|$$

$$\nu = \frac{c}{\lambda} \Rightarrow \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$\therefore u(\lambda) = \frac{8\pi\nu^2 k_B T}{c^3}\frac{c}{\lambda^2} = \frac{8\pi\, c^2 k_B T}{c^3\lambda^2}\frac{c}{\lambda^2}$$

$$= \frac{8\pi\, k_B \lambda T}{\lambda^5}$$

$$= \frac{f(\lambda T)}{\lambda^5} \quad \text{with} \quad f(\lambda T) = 8\pi\, k_B \lambda T$$

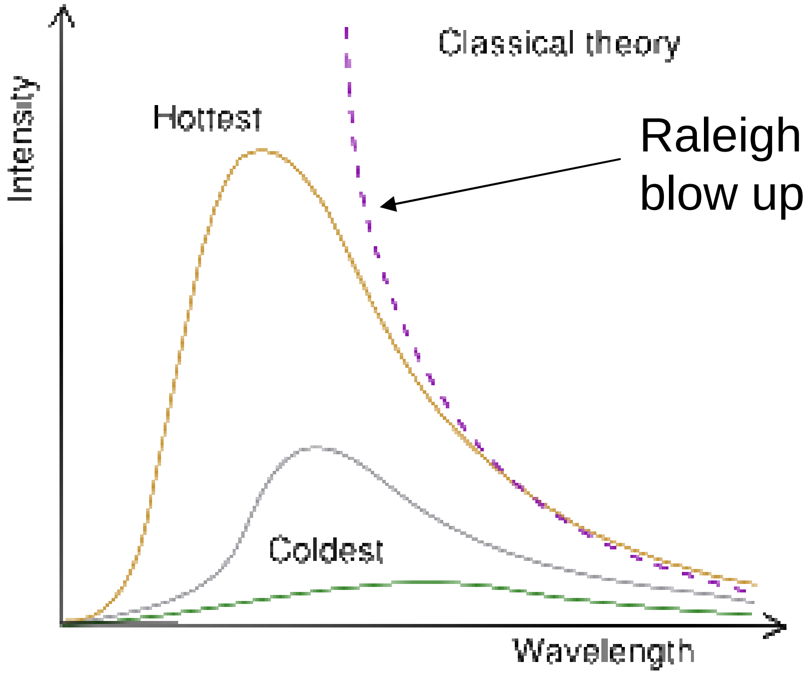


Rayleigh

Wien

Blackbody radiation (Text 2.2)

But don’t be happy so soon!



Raleigh-Jeans formula
blow up as $\nu \to \infty$!

Blackbody radiation (Text 2.2)

Those density of state stuff is just simple counting and arithmetic, and I can’t see anything wrong in it.

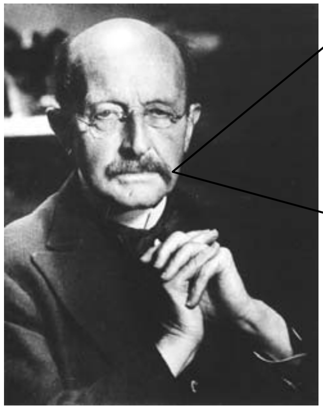
To make the theoretical curve fit the experimental curve, the energy of the standing waves has to depend on ν. In other wards, the equipartition law has to be discarded.

After trying many times, I find

$$E(\nu) = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1} \qquad \text{(h is a constant)}$$

will produce a perfect match between theoretical curve and experimental data.

Why don’t you call the constant h under my name?



Max Planck

$h = 6.626\times10^{-34}$ Js

Blackbody radiation (Text 2.2)

Following Planck's suggestion :

$$u(\nu)d\nu V = E(\nu)d\nu\, V \Rightarrow u(\nu) = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1}\frac{8\pi\nu^2}{c^3} \Rightarrow u(\nu) = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{\frac{h\nu}{k_B T}} - 1} \quad \text{(Eq. (2.4 in text))}$$

Although the final result is correct, but the way we present the derivation is kind of misleading and actually wrong (after we know the right physics). The proper way to do it is:

$$\therefore \quad u(\nu)d\nu V = n(\nu)G(\nu)E(\nu)d\nu\, V$$

$$\Rightarrow u(\nu) = \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}\frac{8\pi\nu^2}{c^3}h\nu$$

$$\Rightarrow u(\nu) = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{\frac{h\nu}{k_B T}} - 1} \quad \text{(Eq. (2.4 in text))}$$

In other words, we made two mistakes in the original derivation :

1. We over simplify $n(\nu)E(\nu)$ into a single $E(\nu)$ and ignore the statistical term $n(\nu)$.
2. $E(\nu)$ should be $h\nu$ instead of our more complicated form. (The complicated stuff comes from $n(\nu)$.)

This equation match experimental data perfectly!

Note that this equation also follow Wien's Law!

Wien

One mistake will make things wrong, but two mistakes may make things correct!