Kittel (7th Edition). Chapter 2. Problem 1.

(a)

Let the planes intercepts the base vectors $\vec{a}_1, \vec{a}_2$, and $\vec{a}_3$ at $p\vec{a}_1, q\vec{a}_2$, and $r\vec{a}_3$.

If we have to multply the reciprocals of p, q, and r with a common factor s to form the Miller indices, i.e.

$$s\left(\frac{1}{p}\frac{1}{r}\frac{1}{s}\right) = (h\ k\ l) \Rightarrow p = \frac{s}{h}\ ,\ q = \frac{s}{k},\ r = \frac{s}{l}$$

Note : 1. There is always a lattice plane passing through the origin

2. s used to multiply 1/p, 1/q, and 1/r is an integer

3. The reason why $s \neq 1$ is because the p, q, r chosen is not the plane closest to the origin. $s = ..., -3, -2, -1, 0, 1, 2, 3, ...$ correspond to consecutive lattice planes (all have the same Miller indices hkl). $s = 0$ is the plane passing through the origin, $s = \pm 1$ are the nearest planes at either side of the $s = 0$ plane and so on.

In the derivation below we can simply choose $s = 1$, but we keep it there until the end.

The plane is generared by vecrors $(p\vec{a}_1 - q\vec{a}_2)$ and $(p\vec{a}_1 - r\vec{a}_3)$, hence we need only to show these two vectors are perpendicular to $\vec{G}$:

$$\begin{aligned}
\vec{G}\cdot(p\vec{a}_1 - q\vec{a}_2) &= (h\vec{b}_1 + k\vec{b}_2 + l\vec{b}_3)\cdot(p\vec{a}_1 - q\vec{a}_2) \\
&= (h\vec{b}_1 + k\vec{b}_2)\cdot(\frac{s}{h}\vec{a}_1 - \frac{s}{k}\vec{a}_2) \qquad (\vec{b}_3 \perp \vec{a}_1 \text{ and } \vec{a}_2) \\
&= h\vec{b}_1\cdot\frac{s}{h}\vec{a}_1 - k\vec{b}_2\cdot\frac{s}{k}\vec{a}_2 \qquad (\vec{b}_1 \perp \vec{a}_2 \text{ and } \vec{b}_2 \perp \vec{a}_1) \\
&= \frac{2\pi s}{V}[(\vec{a}_2\times\vec{a}_3)\cdot\vec{a}_1 - (\vec{a}_3\times\vec{a}_1)\cdot\vec{a}_2] \\
&= \frac{2\pi s}{V}[(\vec{a}_3\times\vec{a}_1)\cdot\vec{a}_2 - (\vec{a}_3\times\vec{a}_1)\cdot\vec{a}_2] = 0
\end{aligned}$$

$$\begin{aligned}
\vec{G}\cdot(p\vec{a}_1 - r\vec{a}_3) &= (h\vec{b}_1 + k\vec{b}_2 + l\vec{b}_3)\cdot(p\vec{a}_1 - r\vec{a}_3) \\
&= (h\vec{b}_1 + l\vec{b}_3)\cdot(\frac{s}{h}\vec{a}_1 - \frac{s}{l}\vec{a}_3) \qquad (\vec{b}_2 \perp \vec{a}_1 \text{ and } \vec{a}_3) \\
&= h\vec{b}_1\cdot\frac{s}{h}\vec{a}_1 - l\vec{b}_3\cdot\frac{s}{l}\vec{a}_3 \qquad (\vec{b}_1 \perp \vec{a}_3 \text{ and } \vec{b}_3 \perp \vec{a}_1) \\
&= \frac{2\pi s}{V}[(\vec{a}_2\times\vec{a}_3)\cdot\vec{a}_1 - (\vec{a}_1\times\vec{a}_2)\cdot\vec{a}_3] \\
&= \frac{2\pi s}{V}[(\vec{a}_1\times\vec{a}_2)\cdot\vec{a}_3 - (\vec{a}_1\times\vec{a}_2)\cdot\vec{a}_3] = 0
\end{aligned}$$

(b)

Suppose we have to multiply $\vec{G}$ with a factor t so that the tip $t\vec{G}$ is on the plane :

$$(p\vec{a}_1 - t\vec{G}) \cdot \vec{G} = 0 \qquad \text{(we have already proved that } \vec{G} \perp \text{plane)}$$

$$\Rightarrow \quad p\vec{a}_1 \cdot \vec{G} = tG^2$$

$$\Rightarrow \quad p\vec{a}_1 \cdot h\vec{b}_1 = tG^2$$

$$\Rightarrow \quad s\,\vec{a}_1 \cdot \left[\frac{2\pi \vec{a}_2 \times \vec{a}_2}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)} \right] = tG^2$$

$$\Rightarrow \quad t = \frac{2\pi s}{G^2}$$

$$\therefore \left| t\vec{G} \right| = \frac{2\pi s}{G^2} G = \frac{2\pi s}{G}$$

Distance between planes = Distance between the (s = 1) plane and the origin

$$= \frac{2\pi}{G}$$

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(c)

Reciprocal lattice of a simple cubic of lattica parameter a is again a simple cubic of lattice parameter $\frac{2\pi}{a}$. $\therefore \vec{b}_1, \vec{b}_2$, and $\vec{b}_3$ are perpendicular to each other and each has a magnitude of $\frac{2\pi}{a}$.

$$\therefore G = \sqrt{\left(\frac{2\pi h}{a}\right)^2 + \left(\frac{2\pi k}{a}\right)^2 + \left(\frac{2\pi l}{a}\right)^2} = \frac{2\pi}{a}\sqrt{h^2 + k^2 + l^2}$$

$$d(hkl) = \frac{2\pi}{G} = \frac{a}{\sqrt{h^2 + k^2 + l^2}} \quad \text{or} \quad d^2 = \frac{a^2}{h^2 + k^2 + l^2}$$