Kittel (8th Edition). Chapter 3. Problem 5.

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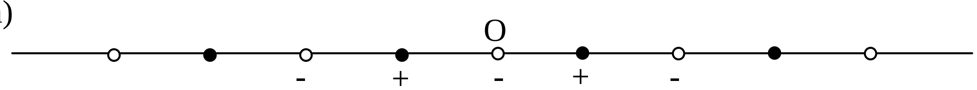
(a)



$$
\begin{aligned}
U_{tot} &= N \times \text{potential energy correspondes to one single ion} \\
&= N \sum_{j \neq 0} U_{0j} \\
&= 2N\left[2\frac{A}{R^n} + \sum_{j \neq 0} \pm \frac{q^2}{Rp_{0j}} \right] \\
&= 2N\left[2\frac{A}{R^n} + \frac{q^2}{R} \sum_{j \neq 0} \pm \frac{1}{p_{0j}} \right] \\
&= 2N\left[2\frac{A}{R^n} - \frac{q^2\alpha}{R} \right]
\end{aligned}
$$

where $\alpha = \sum_{j \neq 0} \mp \frac{1}{p_{0j}}$ is the Madelung constant.

$$
\begin{aligned}
\alpha = \sum_{j \neq 0} \mp \frac{1}{p_{0j}} &= (......+\frac{1}{3}-\frac{1}{2}+\frac{1}{1}+\frac{1}{1}-\frac{1}{2}+\frac{1}{3}+....) \\
&= 2(1-\frac{1}{2}+\frac{1}{3}+....) \\
&= 2\ell n2
\end{aligned}
$$

because $\ell n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ...$

$$
\therefore \quad U_{tot} = 2N\left[2\frac{A}{R^n} - \frac{q^2 2\ell n2}{R} \right] = N\left[\frac{4A}{R^n} - \frac{2q^2\ell n2}{R} \right]
$$

$$
\left.\frac{\partial U_{tot}}{\partial R}\right|_{R=R_0} = 0 \Rightarrow -\frac{4nA}{R_0^{\,n+1}} + \frac{2q^2\ell n2}{R_0^{\,2}} = 0 \Rightarrow \frac{4A}{R_0^{\,n}} = \frac{2q^2\ell n2}{nR_0}
$$

$$
\therefore \quad U_{tot}(R_0) = N\left[\frac{2q^2\ell n2}{nR_0} - \frac{2q^2\ell n2}{R} \right] = \underline{\underline{-\frac{2Nq^2\ell n2}{R_0}\left[1-\frac{1}{n}\right]}}
$$

(b) If the compression is adiabatic, dW=dU.

$$U_{tot,} = N\left[\frac{4A}{R^n} - \frac{2q^2\ell n2}{R}\right]$$

$$R \to R\,(1-\delta)$$

$$U_{tot,} = N\left[\frac{4A}{[R\,(1-\delta)]^n} - \frac{2q^2\ell n2}{[R\,(1-\delta)]}\right]$$

$$\approx N\left[\frac{4A}{R^n}(1+(-n)(-\delta)+\frac{(-n)(-n-1)}{2}(-\delta)^2+\cdots) - \frac{2q^2\ell n2}{\pi\varepsilon_0 R}(1+(-1)(-\delta)+\frac{(-1)(-1-1)}{2}\delta^2+\cdots)\right]$$

$$= \underbrace{N\left[\frac{4A}{R^n} - \frac{2q^2\ell n2}{R}\right]}_{U_{Tot}\text{ when } R=R_0} + \underbrace{N\left[\frac{4nA}{R^n} - \frac{2q^2\ell n2}{R}\right]\delta}_{=0\text{ when } R=R_0} + N\left[\frac{4A}{R}\frac{n(n+1)}{2} - \frac{2q^2\ell n2}{R}\right]\delta^2$$

$$\therefore \delta U = N\left[-\frac{4A}{R_0{}^n}\frac{n(n+1)}{2} + \frac{2q^2\ell n2}{R_0}\right]\delta^2 = N\left[\frac{2q^2\ell n2}{R_0}\frac{(n+1)}{2} - \frac{2q^2\ell n2}{R_0}\right]\delta^2$$

$$= N\left[\frac{q^2\ell n2}{R_0}(n-1)\right]\delta^2$$

$$\therefore \text{Work done per unit length} = \frac{\delta U}{2NR_0} = \left[\frac{q^2\ell n2}{2R_0{}^2}(n-1)\right]\delta^2 = \frac{1}{2}C\delta^2$$

$$\text{with } C = \frac{q^2\ell n2}{R_0{}^2}(n-1)$$