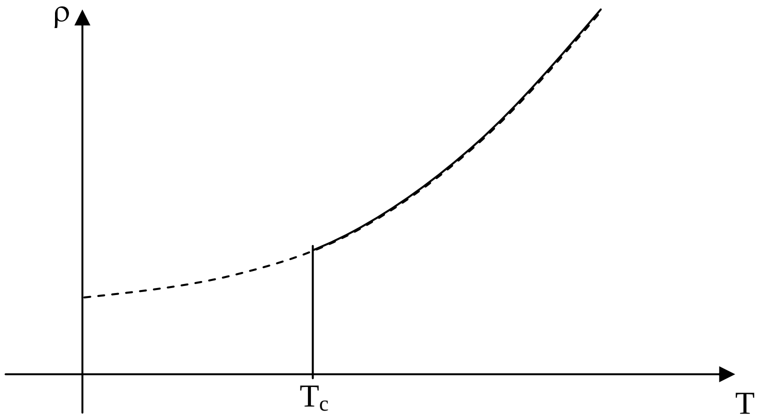
Chapter 10. Superconductivity

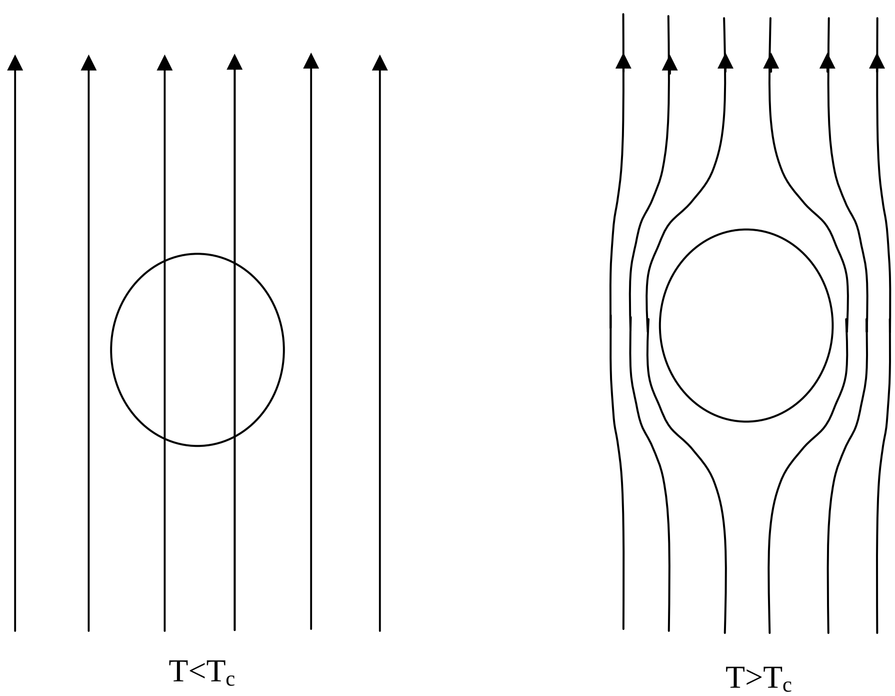
Two basic properties of superconductivity

1. Zero resistivity

Below a certain temperature, the critical temperature T_c (property of the superconductor), resistivity of a superconductor will become *exactly zero*. The first superconductor was mercury, discovered by Onnes in 1911.



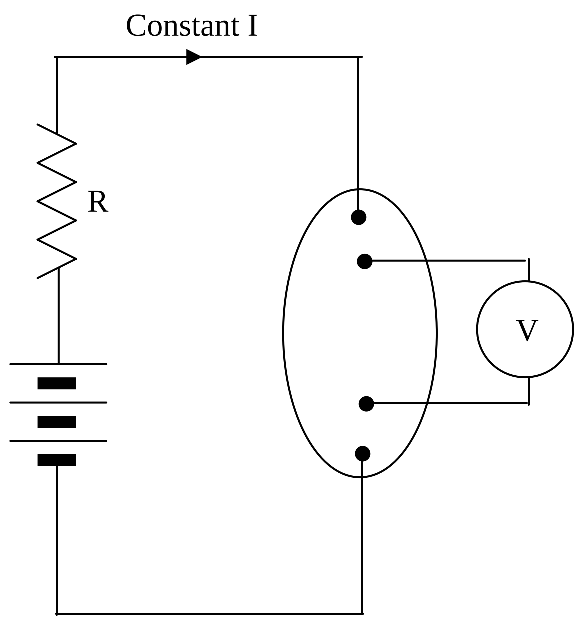
2. Perfect diagmagnetism



A superconductor expels magnetic field *completely* when it is in superconducting phase ($T<T_c$). This phenomenon was discovered by Meissner (and Ochsenfeld) in 1933, so it is called the Meisner effect.

3. Missing any one of these two properties will make the superconducting phase *thermodynamically unstable*. Hence it needs both properties to prove a material is a superconductor.

4. In measuring resistance of a superconductor, if contact resistance >> normal resistance of the superconductor, strict four points measurement is needed:



For I to be constant, R>>Sample resistance + contact resistance.

Two fluid model

1. Assume there are two kind of carriers – normal and superconducting. In here, let the normal carriers form component 1, and the superconducting carriers form component 2.

2. Superconducting carriers are in a condensed state, they are at the lowest energy state and they carry no entropy.

3. As a result, there is no scattering for the superconducting carriers and there is no resistance for them – they cause the phenomenon of superconductivity.

4. When $T>T_c$, all carriers are normal. When T=0, all carriers will be superconducting. When $0<T< T_c$, $\omega = n_s/N$ will be superconducting and $(1-\omega)$ will be normal. ω can be considered as an order parameter. We want now to determine the value of ω for the equilibrium between the two components.

5.

Define free energy $f = u - TS - \frac{1}{4\pi}BH \Rightarrow df = du - SdT - \frac{1}{4\pi}BdH$

Free energy of the neutral component :

For H = 0 and assume $f_1(0,0) = 0$

$$f_1(T,0) = f_1(0,0) - \int C_{en}dT = -\int \gamma TdT = -\frac{1}{2}\gamma T^2$$

where specific heat per unit volume $= \gamma T$

The superconducting component has no entropy, and because of Meissner effect :

$$f_2(T,0) = \text{constant} = f_2(0,0) = f_1(0,0) - \frac{H_c(0)^2}{8\pi} = -\frac{H_c(0)^2}{8\pi}$$

$\therefore$ Free energy of the superconducting phase is :

$$f_s(T,0) = (1-\omega)\,f_1(T,0) + \omega f_2(T,0)$$
$$= (1-\omega)\left(-\frac{1}{2}\gamma T^2\right) + \omega f_2(0,0)$$
$$= (1-\omega)\left(-\frac{1}{2}\gamma T^2\right) - \omega\frac{H_c(0)^2}{8\pi}$$

To fit the experimental data, Gorter and Casimir replaced (1-ω) in above expression by $(1-\omega)^{1/2}$:

$$f_s(T,0) = \sqrt{(1-\omega)}\left(-\frac{1}{2}\gamma T^2\right) - \omega\frac{H_c(0)^2}{8\pi}$$

At equilibrium,

$$\frac{\partial f_s}{\partial\omega} = 0 \Rightarrow \frac{1}{2\sqrt{1-\omega}}\left(\frac{1}{2}\gamma T^2\right) - \frac{H_c(0)^2}{8\pi} = 0$$

$$\Rightarrow \sqrt{1-\omega} = \frac{\left(\frac{1}{2}\gamma T^2\right)}{\frac{H_c(0)^2}{4\pi}} = \frac{2\pi\gamma T^2}{H_c(0)^2}$$

$$\Rightarrow \omega = 1 - \left(\frac{2\pi\gamma T^2}{H_c(0)^2}\right)^2$$

At $T = T_c$, $\omega = 0$

$$\therefore \left(2\pi\gamma T_c^{\,2}\right) = H_c(0)^2 \Rightarrow \boxed{\omega = 1 - \left(\frac{T}{T_c}\right)^4}$$

$$f_s(T,0) = \sqrt{1-\omega}\left(-\frac{1}{2}\gamma T^2\right) - \omega\frac{H_c(0)^2}{8\pi}$$

$$\text{with } \omega = 1 - \left(\frac{T}{T_c}\right)^4$$

$$\therefore f_s(T,0) = \left(\frac{T}{T_c}\right)^2\left(-\frac{1}{2}\gamma T^2\right) - \left[1-\left(\frac{T}{T_c}\right)^4\right]\frac{H_c(0)^2}{8\pi} = \left(\frac{T}{T_c}\right)^2\left(-\frac{1}{2}\gamma T^2\right) - \left[1-\left(\frac{T}{T_c}\right)^4\right]\frac{2\pi\gamma T_c^{\,2}}{8\pi}$$

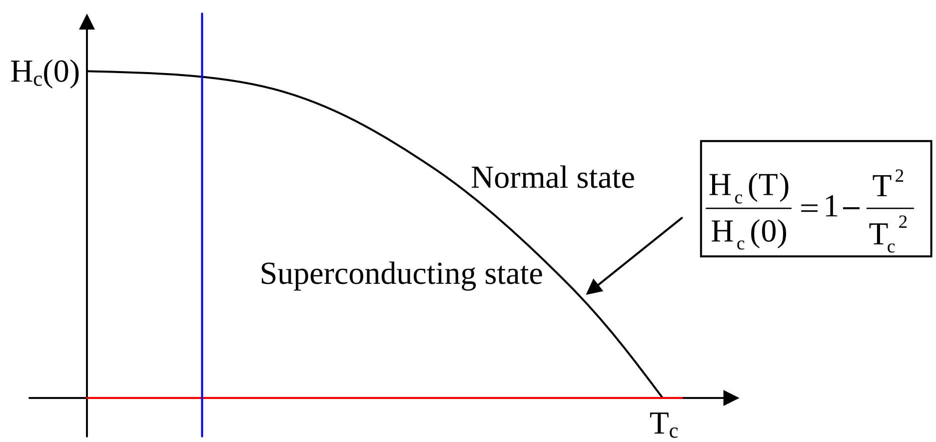
$$= -\frac{1}{2}\frac{\gamma}{T_c^{\,2}}T^4 - \frac{\gamma T_c^{\,2}}{4} + \frac{1}{4}\frac{\gamma}{T_c^{\,2}}T^4$$

$$= -\frac{\gamma}{4}\left(\frac{T^4}{T_c^{\,2}} + T_c^{\,2}\right)$$

Thermodynamics of superconductor

1. Superconducting state is an ordered state, so its free energy and entropy are lower than the free energy and entropy of the normal state.

2. Applied magnetic field can destroy conductivity. A superconducting state will become normal when $H > H_c(T)$.



Define free energy $f=U-TS-HB/4\pi$

3. Along the blue line in the above figure,

$$f_n(T,H) = f_n(T,0) - \frac{H^2}{8\pi}$$

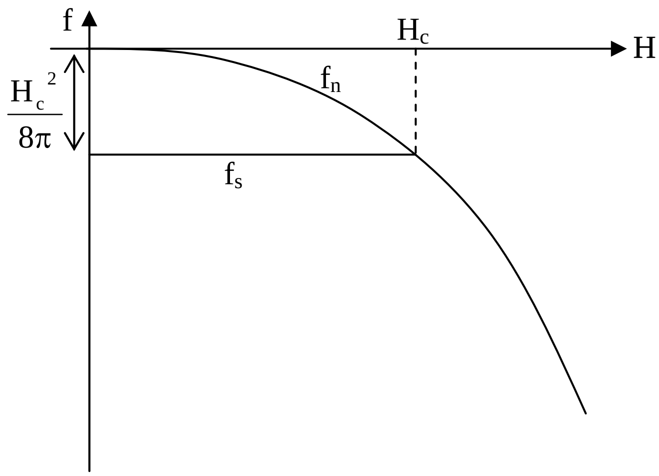
Meissner effect

$$\Rightarrow f_s(T,H) = f_s(T,0)$$

At the phase boundary,

$$f_n(T,H_c) = f_s(T,H_c)$$

$$\therefore \quad f_s(T,0) = f_s(T,H_c) = f_n(T,H_c) = f_n(T,0) - \frac{H_c^{\,2}}{8\pi}$$



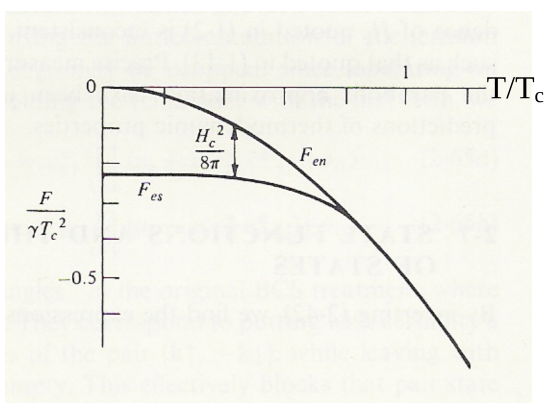
3. Along the red line (T-axis) in above figure:

$$f_n(T,0) = f_n(0,0) - \int C_{en}dT = f_n(0,0) - \int \gamma T dT = f_n(0,0) - \frac{1}{2}\gamma T^2 = -\frac{1}{2}\gamma T^2 \quad (f_n(0,0)=0)$$

where specific heat per unit volume $= \gamma T$

$$f_n(T,0) - f_s(T,0) = \frac{H_c(T)^2}{8\pi} \Rightarrow \boxed{f_s(T,0) = f_n(T,0) - \frac{H_c(T)^2}{8\pi}}$$

$$= f_n(0,0) - \frac{1}{2}\gamma T^2 - \frac{H_c(T)^2}{8\pi} = -\frac{1}{2}\gamma T^2 - \frac{H_c(T)^2}{8\pi}$$



(d)

The two curves have the same slope and join together at $T=T_c$, hence the transition is *second order*.

4. From two fluid model:

$$f_s(T,0) = -\frac{\gamma}{4}\left(\frac{T^4}{T_c^{\ 2}} + T_c^{\ 2}\right)$$

$$\therefore -\frac{\gamma}{4}\left(\frac{T^4}{T_c^{\ 2}} + T_c^{\ 2}\right) = -\frac{1}{2}\gamma T^2 - \frac{H_c(T)^2}{8\pi} \Rightarrow H_c(T)^2 = 2\pi\gamma\left(\frac{T^4}{T_c^{\ 2}} + T_c^{\ 2}\right) - 4\pi\gamma T^2$$

$$\Rightarrow \frac{H_c(T)^2}{2\pi\gamma T_c^{\ 2}} = \left(\frac{T^4}{T_c^{\ 4}} + 1\right) - 2\frac{T^2}{T_c^{\ 2}}$$

$$\Rightarrow \frac{H_c(T)^2}{H_c(0)^2} \approx 1 - 2\frac{T^2}{T_c^{\ 2}}$$

$$\Rightarrow \frac{H_c(T)}{H_c(0)} = 1 - \frac{T^2}{T_c^{\ 2}}$$

I we write normalized quantity $h = \dfrac{H_c(T)}{H_c(0)}$ and $t = \dfrac{T^2}{T_c^{\ 2}}$

we have $h = 1 - t^2$

Electrodynamics of superconductors – London equation

1. Semiclassical equation of motion:

$$\vec{F} = \frac{\partial}{\partial t} m\vec{v} \quad \Rightarrow -e\vec{E} = \frac{m}{n_s e}\frac{\partial}{\partial t}(n_s e\vec{v})$$

$$\Rightarrow \vec{E} = \frac{m}{n_s e^2}\frac{\partial}{\partial t}\vec{J}_s$$

$$\Rightarrow \boxed{\vec{E} = \frac{\partial}{\partial t}(\Lambda\vec{J}_s)} \qquad \text{where } \boxed{\Lambda = \frac{m}{n_s e^2}}$$

2. Faraday's Law:

$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \nabla\times\left[\frac{\partial}{\partial t}\Lambda\vec{J}\right] + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\nabla\times\Lambda\vec{J} + \frac{\vec{B}}{c}\right) = 0$$

$$\Rightarrow \vec{B} = -c\nabla\times\Lambda\vec{J}$$

3. Introduce *penetration depth* λ_L:

$$\Lambda = \frac{m}{n_s e^2} = \frac{4\pi\lambda_L^{\ 2}}{c^2} \Rightarrow \lambda_L = \sqrt{\frac{mc^2}{4\pi n_s e^2}}$$

4. Combine London's second equation with Maxwell equation:

$$\nabla \times \vec{H} - \underbrace{\frac{\partial \vec{D}}{\partial t}}_{=0} = \frac{4\pi \vec{J}}{c} \;\Rightarrow\; \nabla \times \left(\nabla \times \vec{H}\right) = \frac{4\pi}{c}\left(\nabla \times \vec{J}\right)$$

$$\Rightarrow\; \nabla \times \left(\nabla \times \vec{B}\right) = -\frac{4\pi}{c}\frac{\vec{B}}{c\Lambda} \qquad (\text{with } \vec{B} = \vec{H})$$

$$\Rightarrow\; -\nabla^2 \vec{B} = -\frac{4\pi}{c}\frac{\vec{B}}{c\dfrac{4\pi\lambda_L{}^2}{c^2}}$$

$$\Rightarrow \boxed{\nabla^2 \vec{B} = \frac{1}{\lambda_L{}^2}\vec{B}}$$

5. London Gauge
Canonical momentum:

$$\vec{p} = m\vec{v} + \frac{e}{c}\vec{A}$$

For a ground state system,

$$<\vec{p}>= 0 \;\Rightarrow\; <\vec{v}_s> - \frac{e}{mc}\vec{A}$$

$$\therefore\; <\vec{J}_s>= n_s e <\vec{v}>= -\frac{n_s e^2}{mc}\vec{A} = -\frac{1}{\Lambda c}\vec{A}$$

$$\text{Since } \nabla \cdot \vec{J}_s = 0$$

$$\therefore \boxed{\nabla \cdot \vec{A} = 0} \qquad \text{-- London Gauge}$$

6. Original London's equation can also be derived from this result:

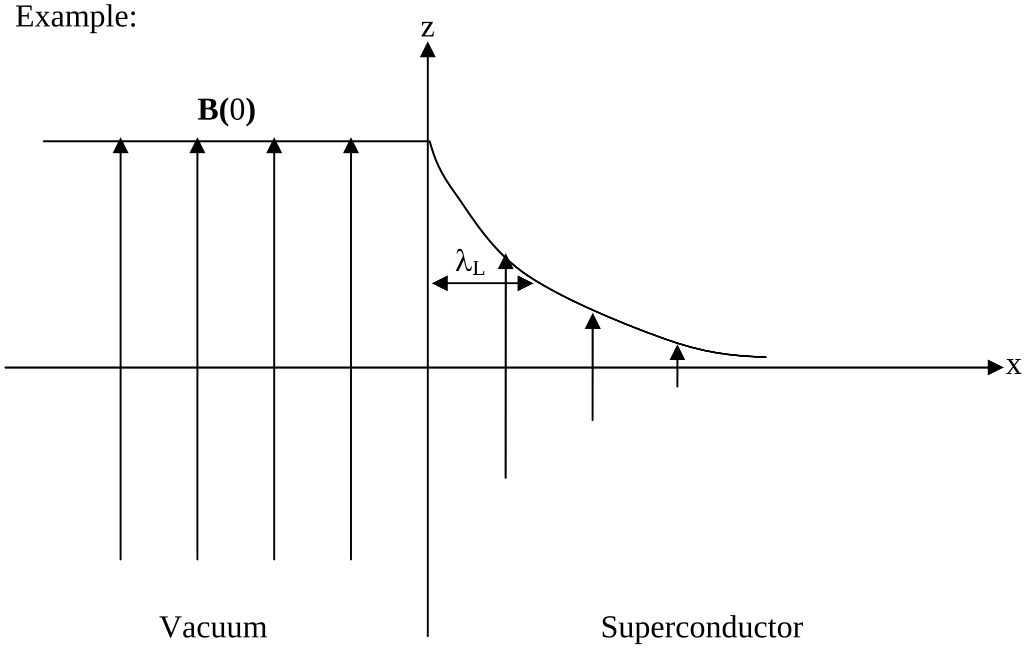
$$\vec{J}_s = -\frac{1}{\Lambda c}\vec{A} \;\Rightarrow\; \frac{\partial}{\partial t}\vec{J}_s = -\frac{1}{\Lambda c}\frac{\partial}{\partial t}\vec{A} \qquad (\text{note that } \frac{1}{c}\frac{\partial}{\partial t}\vec{B} = -\nabla \times \vec{E} \Rightarrow -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} = \vec{E})$$

$$\Rightarrow\; \frac{\partial}{\partial t}\vec{J}_s = -\frac{1}{\Lambda}\vec{E} \qquad (\text{first London'e equation})$$

$$\vec{J}_s = -\frac{1}{\Lambda c}\vec{A} \;\Rightarrow \nabla \times \vec{J}_s = -\frac{1}{\Lambda c}\;\nabla \times \vec{A} \qquad (\vec{B} = \nabla \times \vec{A})$$

$$\Rightarrow \nabla \times \vec{J}_s = -\frac{1}{\Lambda c}\vec{B} \qquad (\text{second London'e equation})$$

7. Example:



$$\nabla^2 \vec{B} = \frac{1}{\lambda_L^{\ 2}} \vec{B} \;\Rightarrow\; \frac{\partial^2}{\partial x^2} B = \frac{1}{\lambda_L^{\ 2}} B$$

$$\Rightarrow B = Ke^{-x/\lambda_L}$$

At x = 0, B = B(0)

$$\therefore B = B(0)e^{-x/\lambda_L}$$

Ginzburg-Landau (GL) equation (1950)

1. Ginzburg-Landau equation is a general phenomenological theory for phase transition by introducing an order parameter Ψ to describe the more ordered state. In the case of superconductor, the superconducting carrier density we used in the two fluid model can be used as the order parameter:

$$n_s = |\Psi|^2$$

2. The GL equation is mostly valid when $T \sim T_c$. Although it is valid only in a short temperature range, but it is extremely powerful when the superconductor becomes inhomogeneous when $\Psi(x)$ is a function of position. Examples when superconductor becomes inhomogeneous: (i) at surface of type I superconductor in an external field, when the field penetrate the superconductor according to London's equations. (ii) when a type II superconductor is in its mixed or vortex state.

3. GL (time independent) equation looks like Schroedinger equation:

$$\frac{1}{2m^*}\left(-i\hbar\nabla - \frac{e^*}{c}\vec{A}\right)^2 \Psi + \underbrace{\beta\,|\Psi^2|\,\Psi}_{\text{Non-linear term}} = -\alpha(T)\Psi$$

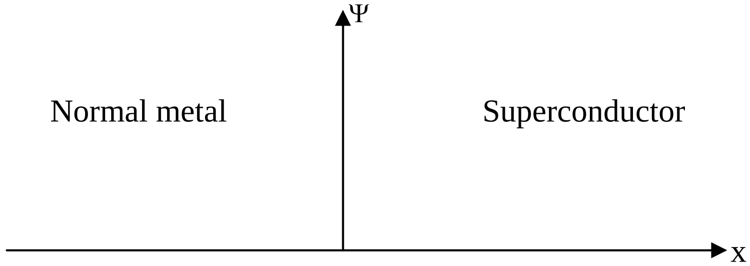
In general, $\alpha<0$ and $\beta>0$. $\Psi(x)$ now plays the role of a wave function and it can be complex.

The supercurrent is given by $\Psi(x)$:

$$J_s = -\frac{ie^* h}{2m^*}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)^2 - \frac{e^{*2}}{m^* c}|\Psi|^2\ \vec{A}$$

4. Experimentally it was determined that e*=2e and m*=2m.

5. Example 1 (proximity effect)



With no magnetic field (**A**=0), GL equation becomes:

$$-\frac{\hbar^2}{2m^*}\nabla^2\Psi + \alpha(T)\Psi + \beta\,|\,\Psi^2\,|\,\Psi = 0$$

Let a = $-\alpha$ and a > 0

$$-\frac{\hbar^2}{2m^*}\frac{d^2}{dx^2}\Psi - a(T)\Psi + \beta\,|\,\Psi^2\,|\,\Psi = 0$$

$$\text{Let } \Psi = \sqrt{\frac{a}{\beta}}\ f$$

$$\therefore \quad -\frac{\hbar^2}{2m^*}\sqrt{\frac{a}{\beta}}\frac{d^2}{dx^2}f - a\sqrt{\frac{a}{\beta}}\Psi + \beta\left(\sqrt{\frac{a}{\beta}}\right)^3 f^3 = 0$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m^* a}\frac{d^2}{dx^2}f - f + f^3 = 0$$

Now introduce *coherence length* ξ_L:

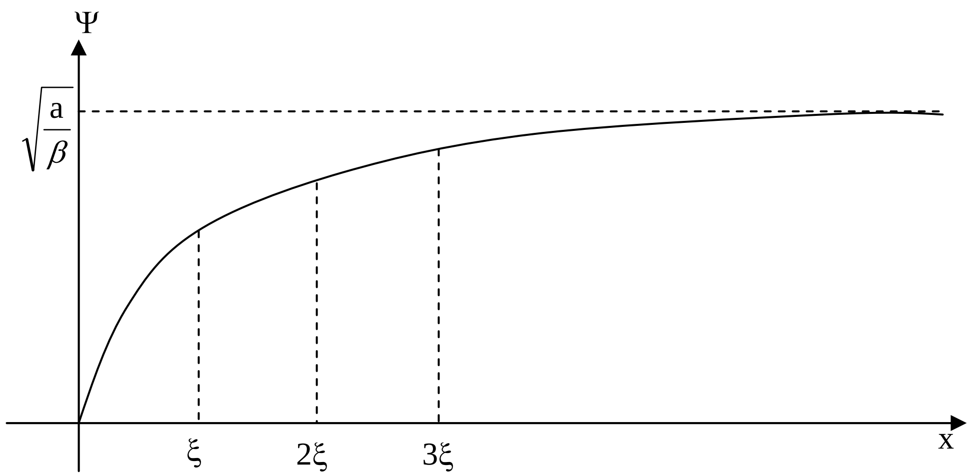
$$\xi_L{}^2 = \frac{\hbar^2}{2m^* a} \Rightarrow \xi_L = \frac{\hbar}{\sqrt{2m^* a}}$$

$$\text{Let } \eta = \frac{x}{\xi_L}$$

$$\therefore \quad \frac{d^2}{dx^2}\Psi + \alpha(T)\Psi - \beta\,|\,\Psi^2\,|\,\Psi = 0$$

$$\text{Solution: } \quad f = \tanh\frac{\eta}{\sqrt{2}}$$

$$\Psi = \sqrt{\frac{a}{\beta}}\ \tanh\frac{x}{\xi\sqrt{2}}$$



6. GL equation introduces a new length scale: coherence length ξ_L. It is a measure on how fast Ψ drops to 0 at the boundary between normal metal and superconductor. ξ_L depends on temperature, because a depends on temperature:

$$\boxed{\xi_L(T) = \frac{\hbar}{\sqrt{2m^* a(T)}} = \frac{\hbar}{|2m^*\alpha(T)|^{\frac{1}{2}}}}$$

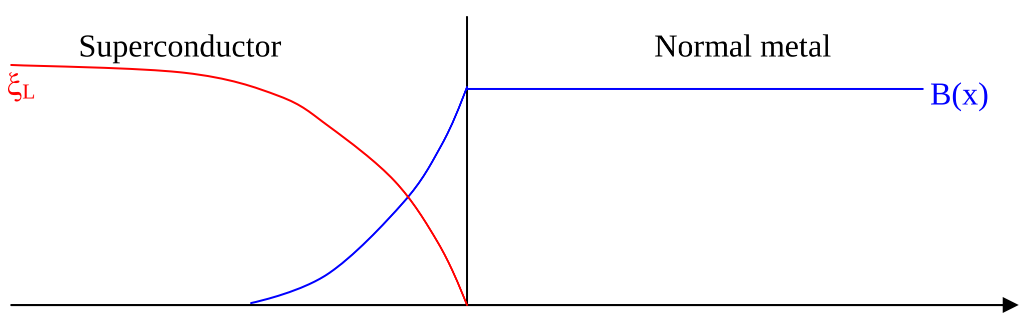
7. α(T) varies linearly with T:

$\alpha(T) \sim T - T_c$

$$\boxed{\therefore \xi_L(T) \sim \frac{1}{\sqrt{T_c - T}}}$$

Note that ξ_L diverges (to infinity) as $T \rightarrow T_c$.

8. Now we have two length scales: penetration depth λ and coherence length ξ_L.



The relative magnitude of these two length scales will affect how the normal region is formed as the magnetic field penetrates the superconductor. For example, if $\lambda >> \xi_L$, one can expect the field can penetrate deep inside the superconductor with the formation of small normal region ($\sim\xi_L$).

9. Define GL parameter κ:

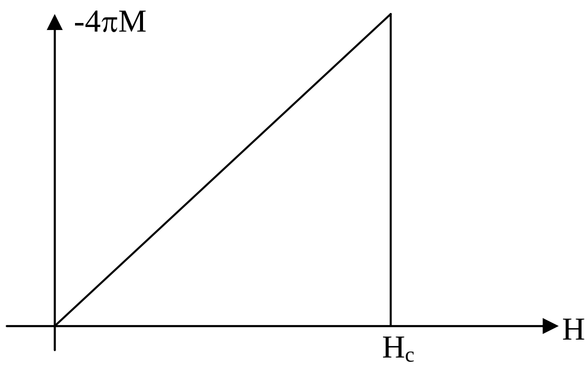
$$\boxed{\kappa = \frac{\lambda}{\xi_L}}$$

According to Abrikosov theory (1957), κ define two types of superconductors:

$$\kappa \begin{cases} < \dfrac{1}{\sqrt{2}} & \text{- Type I superconductor} \\ > \dfrac{1}{\sqrt{2}} & \text{- Type II superconductor} \end{cases}$$

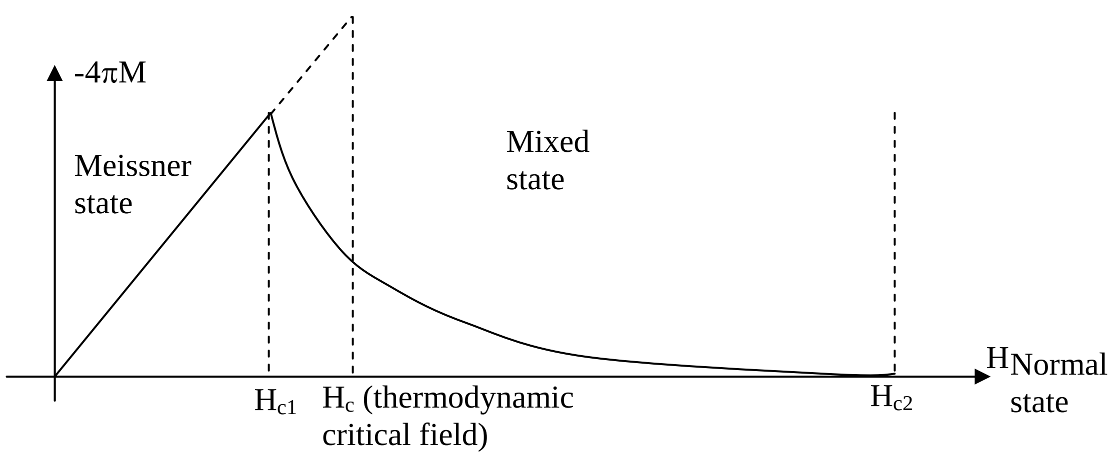
κ determines how the magnetic energy is distributed between surface and volume. For type I superconductor, most energy is stored in the surface. If λ is large, it is more efficient to store energy in "tubes" of ξ_L in diameter. These tubes are called vortices, occur only in type II superconductors.

10. Type-I superconductor

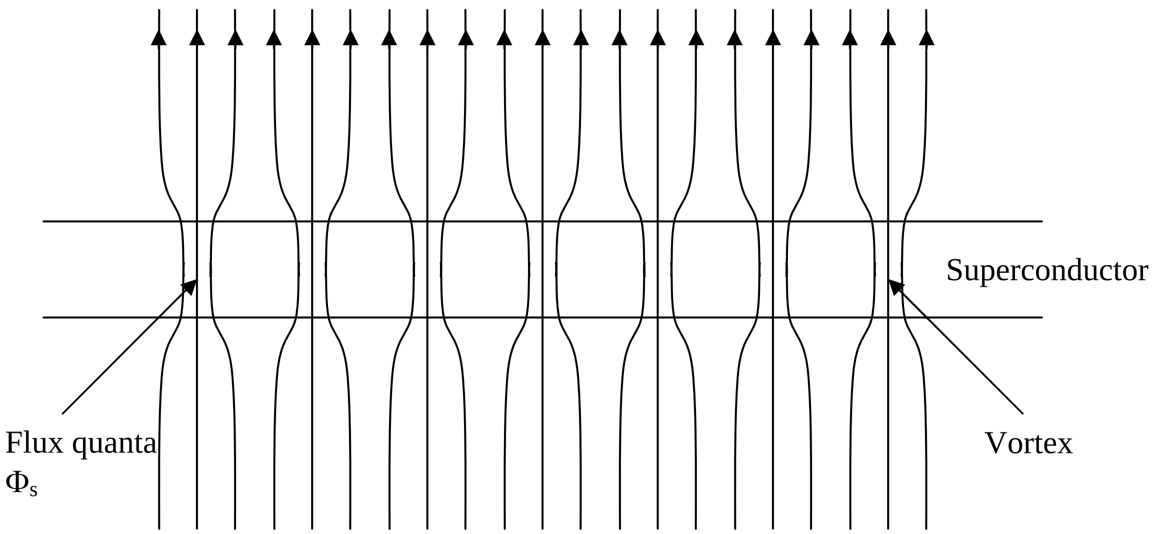


London's equations are followed when it is in the superconducting state ($H < H_c$). Most *elemental* superconductors are type-I superconductors.

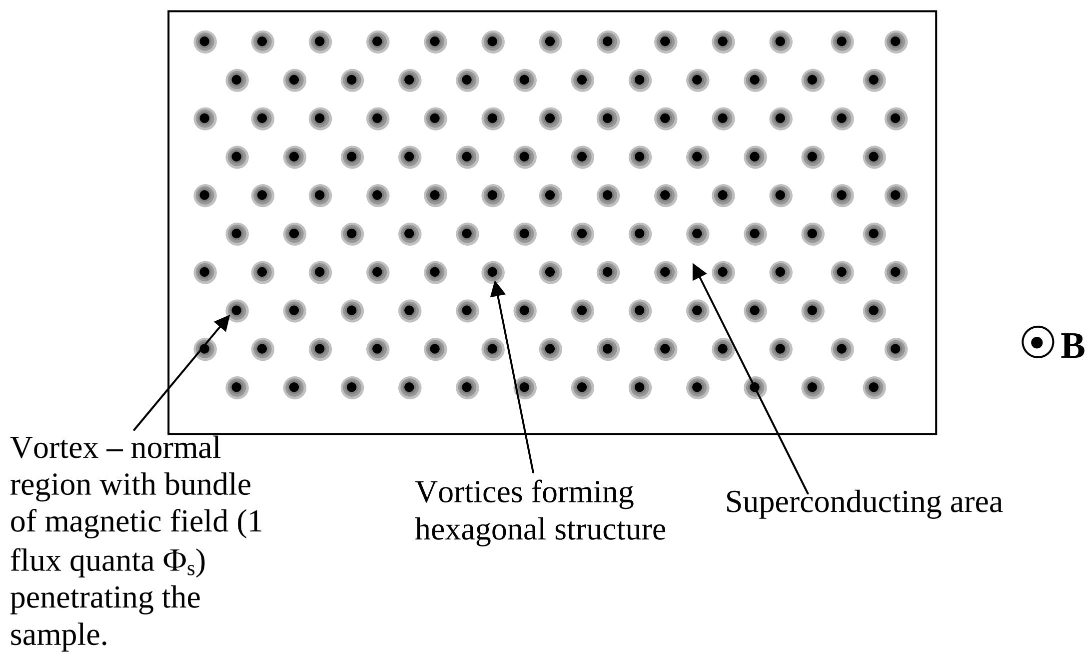
Type-II superconductors.



12. Vortex formation in type II superconductor:



Top view:



13. If we write the order parameter as

$$\Psi = \phi(\bar{r})\, e^{i\theta(\bar{r})}$$

$$\therefore \quad \nabla\Psi = (i\,\phi\nabla\theta + \nabla\phi)\, e^{i\theta}$$

$$\Psi^*\nabla\Psi = i\,\phi^2\nabla\theta + \phi\nabla\phi$$

$$\nabla\Psi^* = (-i\,\phi\nabla\theta + \nabla\phi)\, e^{-i\theta}$$

$$\Psi\nabla\Psi^* = -\,i\,\phi^2\nabla\theta + \phi\nabla\phi$$

$$\vec{J}_s = -\frac{ie^*\hbar}{2m^*}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - \frac{e^{*2}}{m^*c}\left|\Psi\right|^2 \vec{A}$$

$$\Rightarrow \vec{J}_s = \frac{e^*\hbar}{m^*}\left|\phi\right|^2 \nabla\theta - \frac{e^{*2}}{m^*c}\left|\phi\right|^2 \vec{A}$$

$$\Rightarrow \oint \frac{\vec{J}_s}{\left|\phi\right|^2}\cdot d\vec{\ell} = \frac{e^*\hbar}{m^*}\oint \nabla\theta\cdot d\vec{\ell} - \frac{e^{*2}}{m^*c}\oint \vec{A}\cdot d\vec{\ell}$$

Integration of phase θ around a closed path must be a multiple of 2π

i.e. $\oint \nabla\theta\cdot d\vec{\ell} = 2n\pi$

$$\therefore \oint \frac{\vec{J}_s}{\left|\phi\right|^2}\cdot d\vec{\ell} = \frac{2n\pi e^*\hbar}{m^*} - \frac{e^{*2}}{m^*c}\oint \vec{A}\cdot d\vec{\ell}$$

$$\Rightarrow \frac{m^*c}{e^{*2}}\oint \frac{\vec{J}_s}{\left|\phi\right|^2}\cdot d\vec{\ell} = n\underbrace{\left(\frac{ch}{e^*}\right)}_{\Phi_s} - \oint \vec{A}\cdot d\vec{\ell}$$

If $\vec{J}_s = 0$

$$\int \vec{B}\cdot d\vec{\sigma} = \oint \vec{A}\cdot d\vec{\ell} = n\left(\frac{ch}{e^*}\right) = n\Phi_s$$

Φ_s is the flux quanta of a vortex. Note that $\Phi_s = \Phi_0 /2$.

BCS Theory (1957)

1. Since $e^*=2e$ and $m^*=2m$. $\therefore$ electrons form pairs. Pairing allows fermion to form boson (after pairing) and Bose-Eisntein condensation to occur.

2. BCS theory is the only successful microscopic theory so far that explains how the electron pair is formed. Yet there are many exotic superconductors with strange behavior that cannot be fully explained yet.

3. Isotopic effect: the Tc of a superconductor depends on the nucleus mass (of the same element).

Frolich theory $\Rightarrow$ $M^{1/2}Tc$ = constant

This is not necessary precisely correct. In general, we have

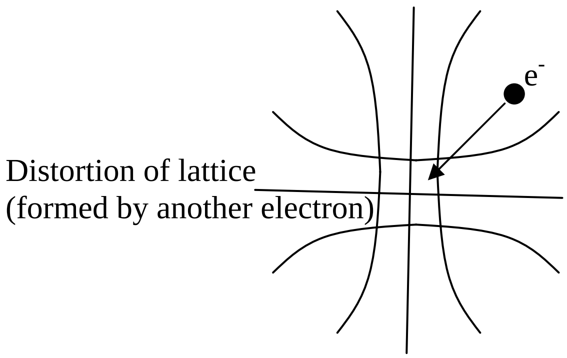
$M^{\alpha}Tc$ = constant

with α to be determined experimentally.

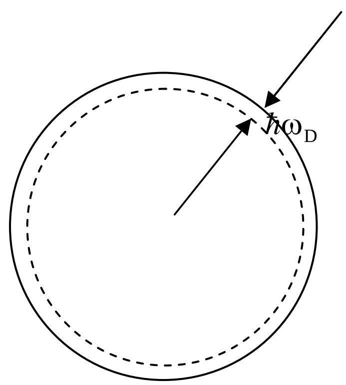
The importance of isotopic effect is that the phenomenon of superconductivity is related to the atoms in the superconductors, not just the free electrons alone.

4. The exchange of virtual phonons produces a small attraction between the electrons. The free electron Fermi sphere is unstable against this small attraction, and a new ground state is defined.

5. . From this, BCS postulated that the electron pairs are mediated by the vibration of atoms (i.e. phonons) – electrons from pairs by exchange of *virtual phonons*. Virtual phonons are phonons with wavelength $\lambda >>$ distance between the electrons forming the pair – this simply means distortion of lattice. The idea is that as an electron moves around in the lattice, its interaction with the ions will cause a small distortion in the lattice and this distortion will attract another electron into the region.



Only electrons within a shell thickness $\hbar\omega_D$ can form pairs. ω_D is the Debye temperature.



5. BCS approximations

(i) For the lowest energy state, electrons form pair so that their total momentum and total spin are zero, i.e. Only consider pairing between electrons with (**k**, ↑) and (-**k**,↓).

(ii) Assume isotropic potential, because of phonon mediated interaction.

$$V_{\vec{k},\vec{k}'} = \begin{cases} -V & \text{if } E_F - \hbar\omega_D \le E_{\vec{k}} \le E_F + \hbar\omega_D \\ 0 & \text{otherwise} \end{cases}$$

This isotropic potential gives rise to "s-wave" superconductors. Meaning of "s-wave" superconductors:

Pairing of two spins.

$$|\uparrow> + |\downarrow> = \begin{cases} \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow> - |\downarrow\uparrow>\right) & \text{s = 0. Singlet state} \Rightarrow \text{Spin part of wavefunction antisymmetric.} \\ |\uparrow\uparrow> & \\ |\downarrow\downarrow> & \text{s = 1. Triplet state} \Rightarrow \text{Spin part of wavefunction symmetric.} \\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow> + |\downarrow\uparrow>\right) & \end{cases}$$

s=0 $\Rightarrow$ symmetric wavefunction.
Angular momentum =s, d, g, i, …….

s=1 $\Rightarrow$ antisymmetric wavefunction.
Angular momentum =p, f, h, …….

Most simple superconductors, as required by BCS theory, are s-wave superconductor. The spatial wave function is isotropic. Pairing of superfluid He3 is p-wave (note that He-4 does not form pair because He-4 is a boson by itself).

7. By forming pairx, the total energy ($2E_F$) is lowered is lowered by an amount of 2Δ. 2Δ is known as the *energy gap* of the superconductor. $\Delta(T)$ can also be considered as the order parameter, related to Ψ.

8. $\Delta(0)$ depends on ω_D and V:

$$\Delta(0) \sim 2\hbar\omega_D e^{-\frac{1}{N(0)V}}$$

N(E=0) is the density of states at the Fermi energy.

9. $\Delta(0)$ determines T_c:

$$\text{For weak coupling superconductor, } \frac{2\Delta(0)}{k_B T} \sim 3.5$$

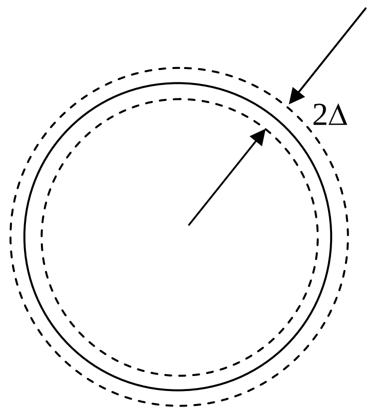
Some superconductors may have a significant larger value. For example, for Ph,

$$\frac{2\Delta_{Pb}(0)}{k_B T} \sim 4.3$$

These are known as strong coupling superconductors. This does *not* mean that these superconductors are not BCS like or phonon mediated.

10. Pippard coherence length ξ_0.

In superconductor, there is another coherence length, called the Pippard coherence length ξ_0. Only electron within 2Δ on the Fermi surface can contribute to superconductivity, even at T=0K.



This corresponds to an uncertainty in time of

$$\delta t \cdot 2\Delta \sim \frac{\hbar}{2} \Rightarrow \delta t = \frac{\hbar}{4\Delta}$$

In this time, the electron will travel a distance of

$$\xi_0 = v_F \delta t = \frac{\hbar v_F}{4\Delta}$$

ξ_0 can be more precisely derived in BCS theory as:

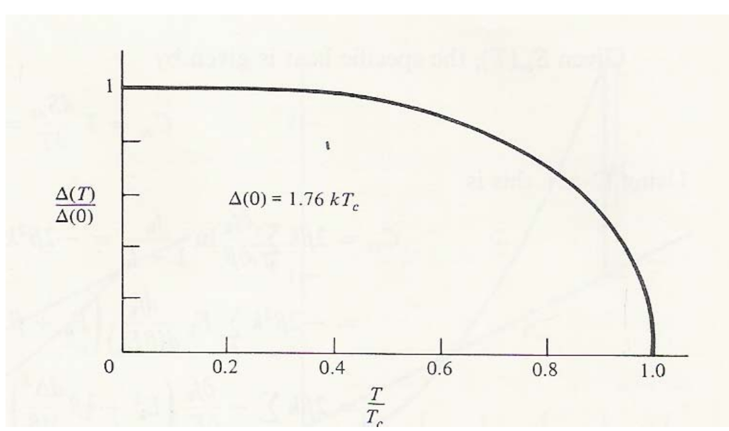
$$\xi_0 = \frac{\hbar v_F}{\pi\Delta}$$

ξ_0 is a constant independent of T, and

$\xi_0 \sim \xi(T)$ (GL coherence length) when $T \ll T_c$.

10. $\Delta(T)$ depends on temperature. Assuming an isotropic Δ, it is given by the equation:

$$\frac{1}{N(0)V} = \int_0^{\hbar\omega_D} \frac{\tanh \frac{1}{2}\beta\sqrt{\zeta^2 + \Delta^2}}{\sqrt{\zeta^2 + \Delta^2}}\, d\zeta \qquad (\beta = \frac{1}{k_B T})$$



As $T \to T_c$, we know that

$$\xi_L(T) \sim \frac{1}{\sqrt{T_c - T}}$$

But

$$\xi_0 = \frac{\hbar v_F}{\pi\Delta}$$

Therefore we expect

$$\Delta(T) \sim \sqrt{T_c - T} \qquad \text{at } T \sim T_c$$

More accurately, according to BCS:

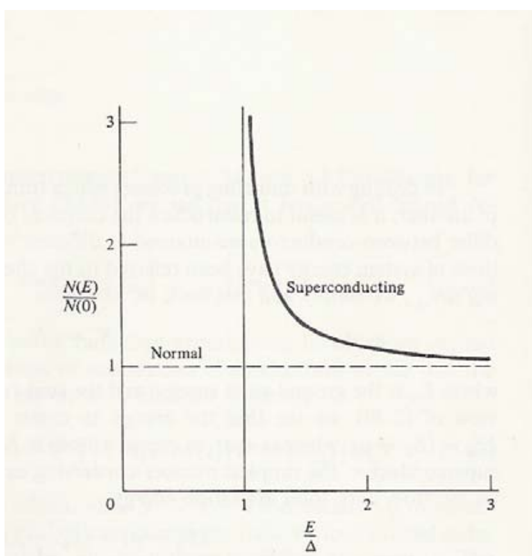
$$\frac{\Delta(T)}{\Delta(0)} \sim \sqrt{1 - \frac{T}{T_c}} \qquad \text{at } T \sim T_c$$

11. Thermal excitations. At T>0, thermal excitations occur. Electrons in a superconductor can be considered as a Fermi liquid. For this reason, thermal excitation can be described as vreation of quasiparticle. A quasiparticle at state **k** ≡ no pair between (**k**, ↑) and (-**k**,↓). Quasiparticles have lifetime. They behave pretty much like an electron at state **k** (Landau Fermi liquid theory). They follow Fermi-Dirac distribution:

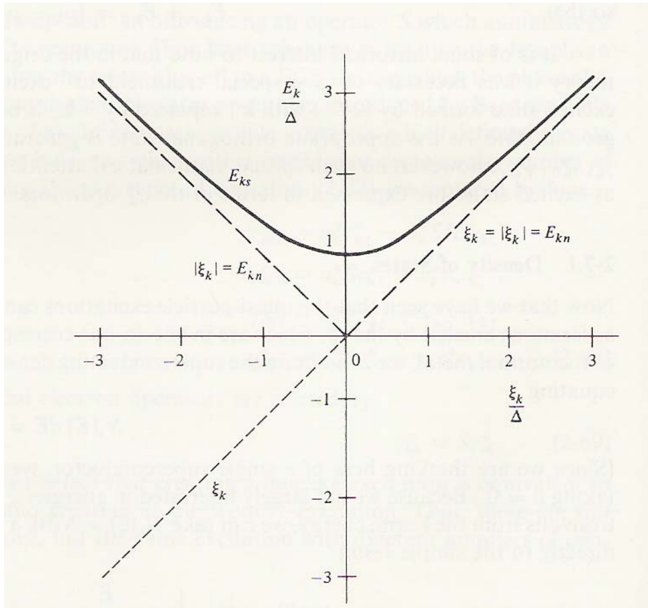
$$f(E_{\vec{k}}) = \frac{1}{e^{E_{\vec{k}}/k_B T} + 1}$$

12. BCS theory gives the density of states of the quasiparticle as:

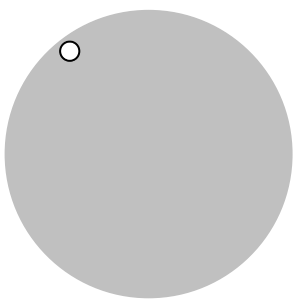
$$\frac{N_S(E)}{N(0)} = \begin{cases} \dfrac{E}{\sqrt{E^2 - \Delta^2}} & (E > \Delta) \\ 0 & (E < \Delta) \end{cases}$$



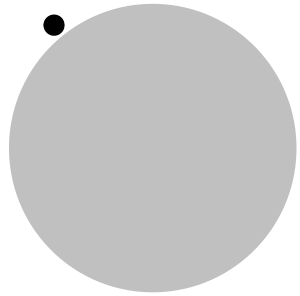
Energy spectrum of quasiparticle:



Hole like quaiparticle:

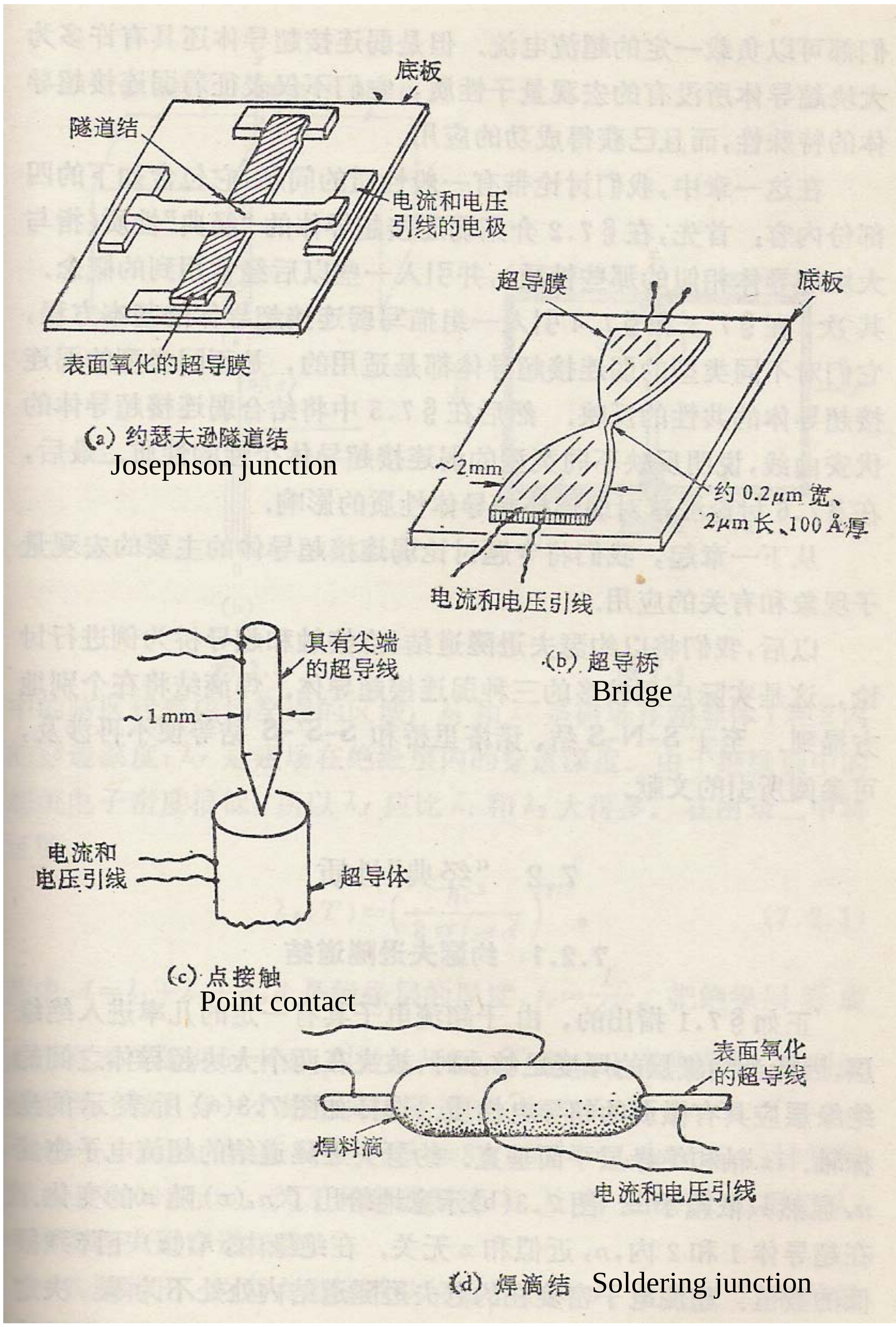


Electron like quasiparticle:



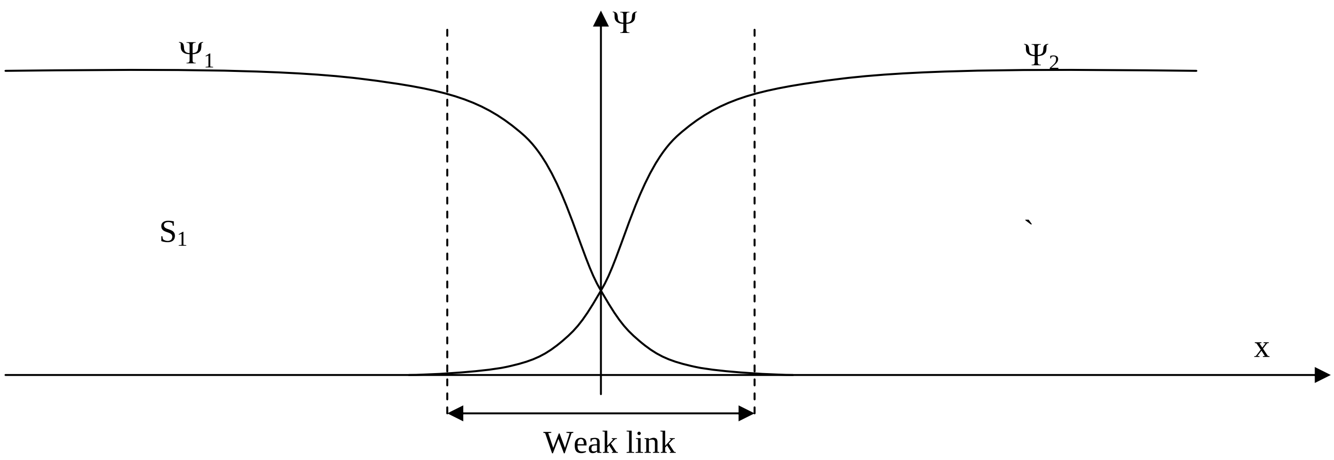
Josephson Tunneling

1. Josephson tunneling is often called a weak link between two superconductors.
Types of weaklinks:



2. dc Josephson effect:

Weak links can conduct supercurrent at *zero* voltage. The critical current of the weak link is much smaller than the bulk critical current.



Time dependent Schroedinger equation:

$$i\hbar\frac{\partial\Psi_1}{\partial t}=H_1\Psi_1 \quad \text{and} \quad i\hbar\frac{\partial\Psi_2}{\partial t}=H_2\Psi_2$$

With coupling by the weak link K:

$$i\hbar\frac{\partial\Psi_1}{\partial t}=H_1\Psi_1+K\Psi_2 \qquad \text{and}$$

$$i\hbar\frac{\partial\Psi_2}{\partial t}=H_2\Psi_2+K\Psi_1$$

If there is a potential difference V across the junction, when a pair "tunnel" through the junction, there is an energy change of 2eV. We can phenomenologically write H_1 as eV and H_2 as –eV at the junction:

$$\therefore\ i\hbar\frac{\partial\Psi_1}{\partial t}=eV\Psi_1+K\Psi_2$$

$$i\hbar\frac{\partial\Psi_2}{\partial t}=-eV\Psi_2+K\Psi_1$$

Since $|\Psi|^2$ =n, we can write

$$\Psi_1=\sqrt{n_1}e^{i\theta_1} \quad \text{and} \quad \Psi_2=\sqrt{n_2}e^{i\theta_2}$$

$$i\hbar\frac{\partial\Psi_1}{\partial t}=eV\Psi_1+K\Psi_2 \quad \Rightarrow \quad i\hbar\left[\frac{1}{2\sqrt{n_1}}e^{i\theta_1}\frac{\partial}{\partial t}n_1+i\sqrt{n_1}e^{i\theta_1}\frac{\partial}{\partial t}\theta_1\right]=eV\sqrt{n_1}e^{i\theta_1}+K\sqrt{n_2}e^{i\theta_2}$$

$$\Rightarrow \quad i\hbar\left[\frac{1}{2}\frac{\partial}{\partial t}n_1+in_1\frac{\partial}{\partial t}\theta_1\right]=eVn_1+K\sqrt{n_1n_2}e^{i(\theta_2-\theta_1)}$$

$$i\hbar\frac{\partial\Psi_2}{\partial t} = -eV\Psi_2 + K\Psi_1 \Rightarrow \quad i\hbar\left[\frac{1}{2\sqrt{n_2}}e^{i\theta_2}\frac{\partial}{\partial t}n_2 + i\sqrt{n_2}e^{i\theta_2}\frac{\partial}{\partial t}\theta_2\right] = -eV\sqrt{n_2}e^{i\theta_2} + K\sqrt{n_1}e^{i\theta_1}$$

$$\Rightarrow \quad i\hbar\left[\frac{1}{2}\frac{\partial}{\partial t}n_2 + in_2\frac{\partial}{\partial t}\theta_1\right] = -eVn_2 + K\sqrt{n_1 n_2}e^{i(\theta_1-\theta_2)}$$

Equating real and imaginary parts:

$$\frac{\hbar}{2}\frac{\partial}{\partial t}n_1 = K\sqrt{n_1 n_2}\sin(\theta_2-\theta_1) \qquad \text{- (1)}$$

$$-\hbar n_1\frac{\partial}{\partial t}\theta_1 = eVn_1 + K\sqrt{n_1 n_2}\cos(\theta_2-\theta_1) \qquad \text{- (2)}$$

$$\frac{\hbar}{2}\frac{\partial}{\partial t}n_2 = K\sqrt{n_1 n_2}\sin(\theta_1-\theta_2) \qquad \text{- (3)}$$

$$-\hbar n_2\frac{\partial}{\partial t}\theta_2 = -eVn_2 + K\sqrt{n_1 n_2}\cos(\theta_1-\theta_2) \qquad \text{- (4)}$$

If $\Delta\theta = \theta_2 - \theta_1$

$$(1) \Rightarrow \frac{\hbar}{2}\frac{\partial}{\partial t}n_1 = K\sqrt{n_1 n_2}\sin\Delta\theta$$

$$(3) \Rightarrow \frac{\hbar}{2}\frac{\partial}{\partial t}n_2 = -K\sqrt{n_1 n_2}\sin\Delta\theta$$

$$\frac{(2)}{\hbar n_1} - \frac{(4)}{\hbar n_2} \Rightarrow \frac{\partial}{\partial t}\Delta\theta = \left[\frac{eV}{\hbar} + \frac{K}{\hbar}\sqrt{\frac{n_2}{n_1}}\cos\Delta\theta\right] - \left[-\frac{eV}{\hbar} + \frac{K}{\hbar}\sqrt{\frac{n_1}{n_2}}\cos\Delta\theta\right]$$

If $n_1 \sim n_2$

$$\boxed{\frac{\partial}{\partial t}\Delta\theta = \frac{2eV}{\hbar}}$$

From (1) and (3) :

$$J = e\frac{d}{dt}(n_1 - n_2) = \frac{2}{\hbar}K\sqrt{n_1 n_2}\sin\Delta\theta - (-K\sqrt{n_1 n_2}\sin\Delta\theta) = \frac{4K\sqrt{n_1 n_2}}{\hbar}\sin\Delta\theta$$

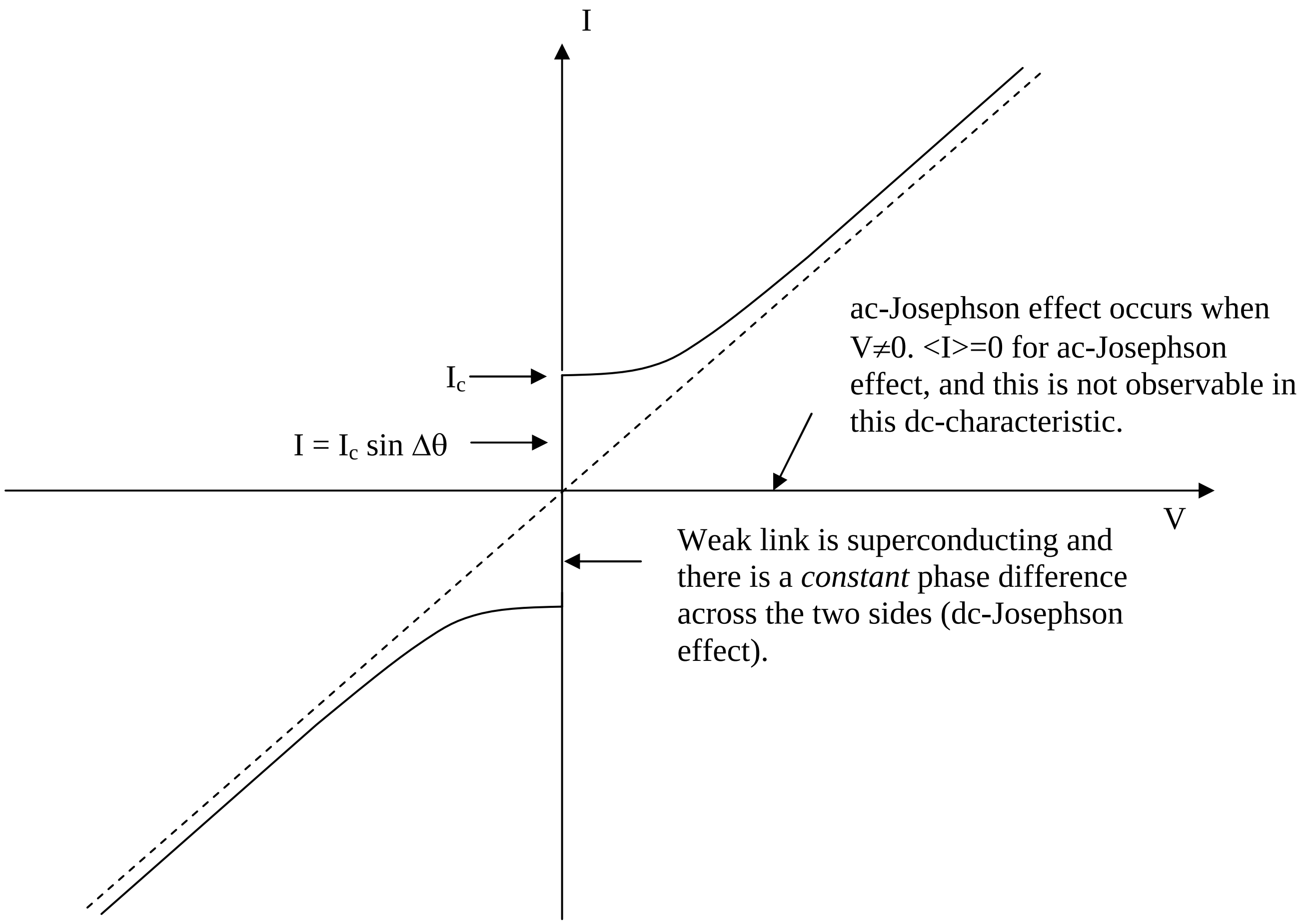
$$\therefore \boxed{J = J_c\sin\Delta\theta} \qquad J_c = \frac{4K\sqrt{n_1 n_2}}{\hbar}$$

In other words,

Josephson super current $\Leftrightarrow$ phase difference across the junction dc-Josephson super current $\Leftrightarrow$ *constant* phase difference across the junction

It is more common to use Josephson current than current density. I=JA, and we can write similarly,

$$I = I_c \sin\Delta\theta$$



3. dc-Josephson effect:

When a *non-zero* voltage is applied across the Josephson junction, the phase difference across the two sides will oscillate and this gives rise to an oscillating supercurrent.

ac-Josephson super current $\Leftrightarrow$ oscillating phase difference across the junction

$$\frac{\partial}{\partial t}\Delta\theta = \frac{2eV}{\hbar} \quad \Rightarrow \Delta\theta = \frac{2eVt}{\hbar} \qquad \text{(constant V)}$$

$$\therefore\ I = I_c \sin\Delta\theta = I_c \sin\frac{2eVt}{\hbar}$$

$$\text{Oscillating frequency} = \omega = \frac{2eV}{\hbar}$$

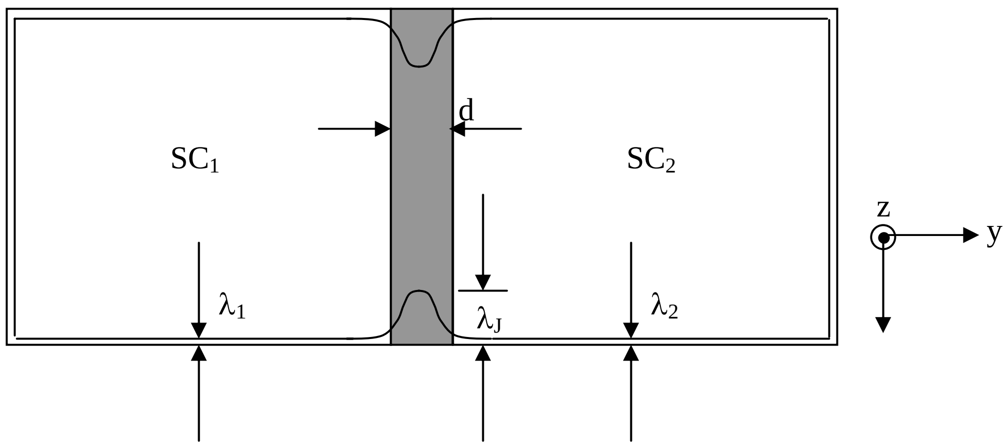
$$\text{If } V = 1\mu V,\ \omega = \frac{2eV}{\hbar} = \frac{2\times 1\times 10^{-6}\times 1.602\times 10^{-19}}{1.05510^{-34}} = 3.037\times 10^{9} = 3.037\ \text{GHz}$$

ac-Josephson is observed as microwave radiation from the junction (caused by the oscillating supercurrent) as a constant voltage is applied across the junction.

reverse, if microwave of frequency ω is shined on a Josephson junction, voltate steps will occur and the step width equals to $\hbar\omega/2e$.

4. Small junction and large junction

When a strong enough magnetic field is applied, it will penetrate the superconductor at the junction area first:

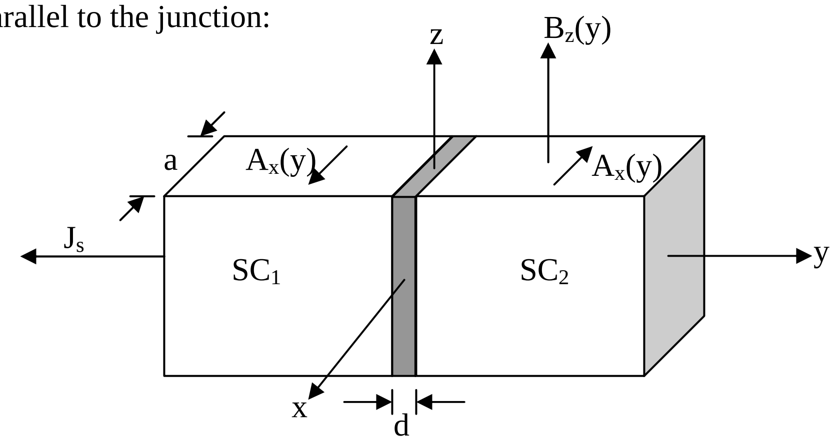


$$\lambda_J(T) = \sqrt{\frac{hc^2}{8\pi J_c e(\lambda_1 + \lambda_2 + d)}}$$

The junction is small if the cross sectional dimensions L_x, $L_x < \lambda_J$. The junction is large if the cross sectional dimensions L_x, $L_x >> \lambda_J$. Below argument applied only to small junctions.

5. Quantum interference

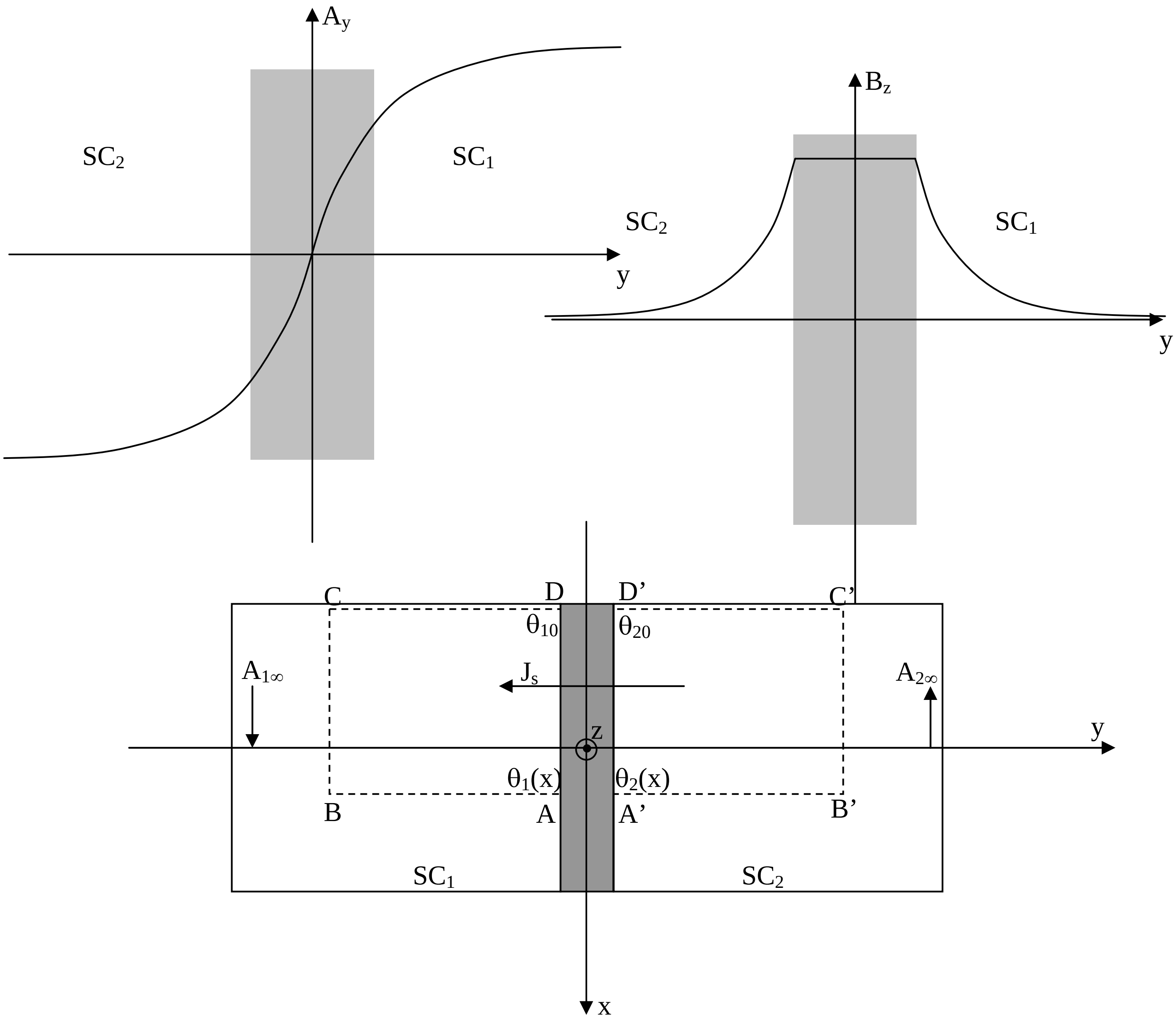
Current density (and phase θ) across the junction will not be uniform if a magnetic field is applied parallel to the junction:



$$\vec{B} = B_z(y)\,\hat{k}$$

$$\vec{B} = \nabla \times \vec{A} \;\Rightarrow\; \vec{A} = -\,yB_0\hat{i} \qquad (|\,y\,| \le \frac{1}{2}d)$$

B_0 is the uniform field in the junction area



For SC_1, along the path ABCD,

$$\frac{e^*\hbar}{m^*}\oint \nabla\theta\cdot d\vec{\ell} = \frac{e^{*2}}{m^*c}\oint \vec{A}\cdot d\vec{\ell} \Rightarrow \oint \vec{A}\cdot d\vec{\ell} = \frac{\hbar c}{e^*}\oint \nabla\theta\cdot d\vec{\ell}$$

$$= \frac{\Phi_s}{2\pi}\oint \nabla\theta\cdot d\vec{\ell}$$

$$= \frac{\Phi_s}{2\pi}\left[\underbrace{\oint_{A\to B} \nabla\theta\cdot d\vec{\ell}}_{=0} + \underbrace{\oint_{B\to C} \nabla\theta\cdot d\vec{\ell}}_{=-A_{1\infty}(x-x_0)} + \underbrace{\oint_{C\to D} \nabla\theta\cdot d\vec{\ell}}_{=0} + \underbrace{\oint_{D\to A} \nabla\theta\cdot d\vec{\ell}}_{=\theta_1(x)-\theta_{10}}\right]$$

For a superconductor, $\vec{B}=0 \Rightarrow \nabla\times\vec{A}=0 \Rightarrow \oint \vec{A}\cdot d\vec{\ell} = 0$

$$\oint \vec{A}\cdot d\vec{\ell} = 0 = 0 \Rightarrow \frac{\Phi_s}{2\pi}[\theta_1(x) - \theta_{10}] = A_{1\infty}(x - x_0)$$

$$\Rightarrow \theta_1(x) = \frac{2\pi}{\Phi_s}A_{1\infty}(x - x_0) + \theta_{10}$$

Similarly, for SC_2, along the path A’B’C’D’,

$$\theta_2(x) = -\frac{2\pi}{\Phi_s}A_{2\infty}(x - x_0) + \theta_{20}$$

Subtracting these two equations,

$$\Delta\theta(x) = \theta_1(x) - \theta_2(x) = \left[\frac{2\pi}{\Phi_s}A_{1\infty}(x - x_0) + \theta_{10}\right] - \left[-\frac{2\pi}{\Phi_s}A_{2\infty}(x - x_0) + \theta_{20}\right]$$

$$= \phi_0 + \frac{2\pi}{\Phi_s}\left(A_{1\infty} + A_{2\infty}\right)x$$

$$\phi_0 = \left(\theta_{10} - \theta_{20}\right) - \frac{2\pi}{\Phi_s}\left(A_{1\infty} + A_{2\infty}\right)x_0$$

$A_{1\infty} + A_{2\infty}$ can be calculated from the combined loop:

$$\Phi = \oint \vec{A}\cdot d\ell = (A_{1\infty} + A_{2\infty})a \Rightarrow (A_{1\infty} + A_{2\infty}) = \frac{\Phi}{a}$$

$$\therefore \Delta\theta(x) = \phi_0 + \frac{2\pi}{\Phi_s}\left(\frac{\Phi}{a}\right)x = \phi_0 + \frac{2\pi}{a}\left(\frac{\Phi}{\Phi_s}\right)x$$

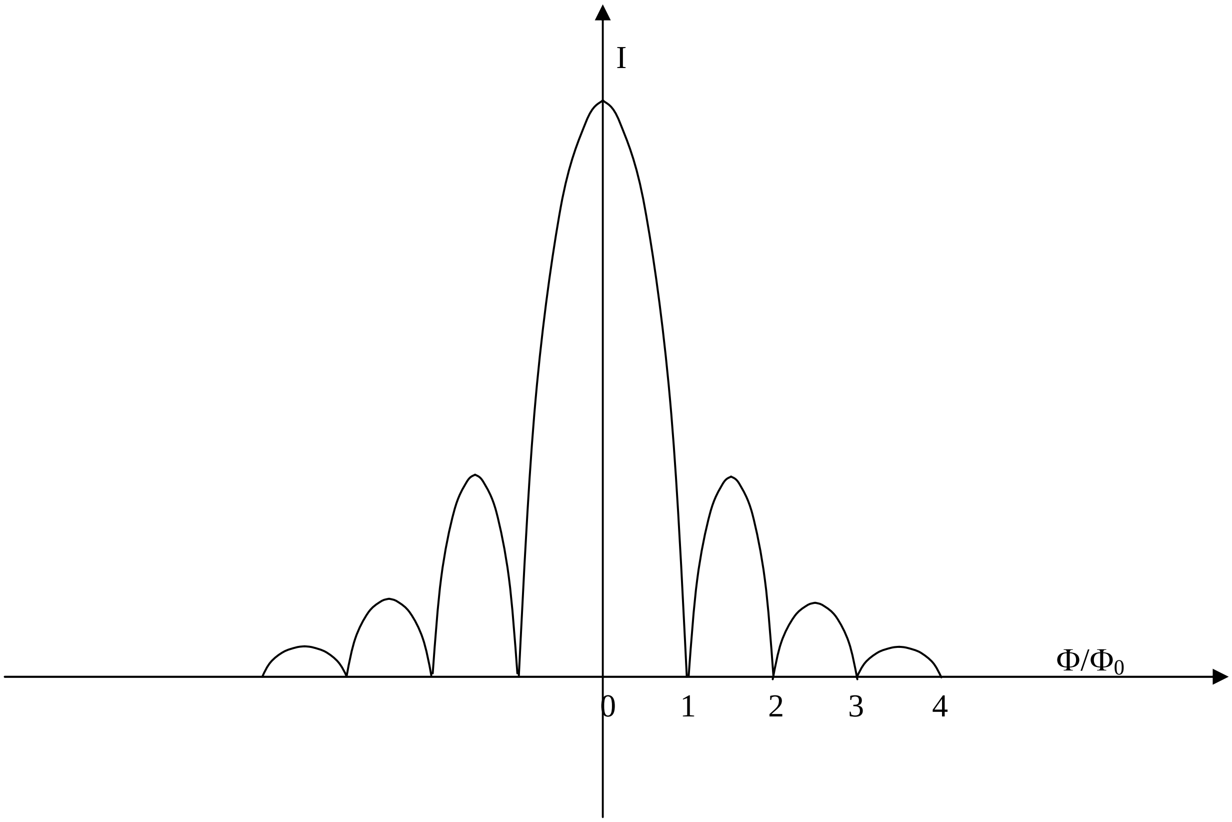
Now

$$J = J_c \sin\Delta\theta(x)$$

$$\therefore \quad I_c(\Phi) = J_c\int \sin\Delta\theta(x)\,dx\,dz$$

$$= \boxed{I_c(0)\frac{\sin\left(\frac{\pi\Phi}{\Phi_s}\right)}{\frac{\pi\Phi}{\Phi_s}}}$$

This gives rise to the Fraunhofer pattern for Josephson junction:



6. SQUID

From above we know that measurement of I_c allows accurate determination of Φ (like using interference to determine small distance). The device using this method to measure Φ is known as Superconducting QUantum Interference Device (SQUID). There are two types of SQUID: ac-SQUID (or rf-SQUID) and dc-SQUID.