Chapter 9. Electrons in magnetic fields

I. Flux quantization

1. Quantization of angular momentum gives rise to quantization of magnetic field.

2. If a charge particle is moving in a close orbit, quantization condition is given by the Bohr-Sommerfeld relation:

$$\oint \vec{p}\cdot d\vec{r} = (n+\frac{1}{2})2\pi\hbar$$

where p is the total momentum of the free electron.

3. In cgs units, if the particle is moving in a magnetic field, **p** is the canonical momentum:

$$\vec{p} = \hbar\vec{k} + \frac{q}{c}\vec{A}$$

4.

$$\therefore \quad \oint \hbar\vec{k}\cdot d\vec{r} + \oint \frac{q}{c}\vec{A}\cdot d\vec{r} = (n+\frac{1}{2})2\pi\hbar$$

$$\Rightarrow \quad \oint \hbar\frac{q}{\hbar c}(\vec{r}\times\vec{B})\cdot d\vec{r} + \oint \frac{q}{c}\nabla\times\vec{A}\cdot d\vec{\sigma} = (n+\frac{1}{2})2\pi\hbar$$

$$\Rightarrow \quad \frac{q}{c}\oint (-\vec{B})\cdot(\vec{r}\times d\vec{r}) + \oint \frac{q}{c}\nabla\times\vec{A}\cdot d\vec{\sigma} = (n+\frac{1}{2})2\pi\hbar$$

$$\Rightarrow \quad -\frac{qB}{c}\underbrace{\oint (\vec{r}\times d\vec{r})}_{2\times\text{Area of orbit}} + \frac{q}{c}\Phi = (n+\frac{1}{2})2\pi\hbar$$

Consider the projection of the closed path on the plane perpendicular to the B - field.

$$\therefore \quad -\frac{2q}{c}\Phi + \frac{q}{c}\Phi = (n+\frac{1}{2})2\pi\hbar \Rightarrow -\frac{q}{c}\Phi = (n+\frac{1}{2})2\pi\hbar$$

Magnetic flux is quantized : $\Phi_0 = \frac{2\pi\hbar c}{e} = \frac{hc}{e} = 4.14\times10^{-7}$ Gauss cm^2 (ot $Tm^{2)}$

II. Motion of electron in a magnetic field

1. In a constant homogeneous field, *classically* in real space, the electron moves in circular loop (or "helix") with cyclotron frequency

$$\omega_c = \frac{qB}{m}$$

Note that ωc does not depend on the radius of the orbit. It depends only on B. The radius (hence area) can be any value and v is proportion to r:

$$v = r\omega_c = \frac{qB}{m}r$$

To maintain the same v, r decreases as B is increased. K is also proportion to r:
Area of the circular loop in real space is proportional (in the same field) to the area of the circular loop in k space:

$$\hbar k = mv \Rightarrow k = \frac{qB}{\hbar} r$$

$$\frac{A_k}{A_r} = \frac{k^2}{r^2} = \left(\frac{qB}{\hbar}\right)^2$$

Note that the ratio depends on B (~B^2). As B increases, A_r decreases (confinement) and A_k increases.

2. Quantum mechanically, r takes only the "quantized radii": Similar to the classical case (but for different reasons), if we track the real orbit of quantum number n, we will find *the orbit shrink as B is increased*. In contrary to classical mechanics, *v increases with B* because v is proportional to r:

$$(A_r)_n B = n\Phi_0 \Rightarrow (A_r)_n = \frac{n\Phi_0}{B}$$

$$\Rightarrow r_n^2 = \frac{n\Phi_0}{\pi B}$$

$$r_n^2 = \frac{n\Phi_0}{\pi B} \Rightarrow v_n^2 = \frac{n\Phi_0}{\pi B}\omega_c^2 = \frac{n\Phi_0 q^2 B}{\pi m^2}$$

$$\text{But } \Phi_0 = \frac{2\pi\hbar}{q}, \therefore v_n^2 = \frac{2nq\hbar B}{m^2}$$

$$k_n^2 = \frac{2nqB}{\hbar}$$

In summary,

Classically we require v=constant because of conservation of energy.

Quantum mechanically, we require quantization of angular momentum or flux quanta.

3. As a result, *the quantized energy of the n-th orbital depends on B*:

$$E_n = \frac{\hbar^2 k_n^2}{2m} + \frac{\hbar^2 k_z^2}{2m} = \frac{\hbar^2 k_z^2}{2m} + \frac{\hbar^2}{2m}\frac{2nqB}{\hbar} = \frac{\hbar^2 k_z^2}{2m} + \frac{n\hbar qB}{m} = \frac{\hbar^2 k_z^2}{2m} + n\hbar\omega_c$$

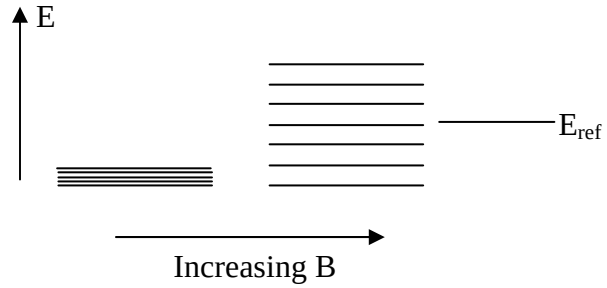
The last term corresponds to the energy of a simple harmonic oscillator of cyclotron frequency ω_c. Indeed, there should be a "zero point energy" correspond to the smallest possible orbital with n=0. i,e, More correctly,

$$E_n = \frac{\hbar^2 k_z^2}{2m} + \left(n + \frac{1}{2}\right)\hbar\omega_c$$

We can also obtain this by solving Schroedinger equation of an electron in a constant B field.

4. In summary,
For the same field:
E, r, and v increase as n increases.

$$r^2 \propto n/B,\ v \propto nB, k \propto nB \text{ and } E \sim n$$



5. Define some arbitrary energy reference E_{ref}. Start from zero field, increase B gradually and energy E_n will increase with B. One by one, all states with $E_n < E_{ref}$ will cross E_{ref}. Define B_n as the B field when $E_n < E_{ref}$.

Also define $(A_k)_{ref}$ as :

$$E_{ref} = \frac{\hbar^2 k_{ref}^{\ 2}}{2m} \Rightarrow E_{ref} = \frac{\hbar^2}{2m\pi}(A_k)_{ref}$$

$$\therefore\ k_n^{\ 2} = \frac{2nqB}{\hbar} \Rightarrow (A_k)_n = \pi k_n^{\ 2} = \frac{2n\pi qB}{\hbar}$$

$$\Rightarrow (A_k)_{ref} = \frac{2n\pi qB_n}{\hbar}$$

$$\Rightarrow \frac{(A_k)_{ref}}{B_n} = \frac{2n\pi q}{\hbar}$$

$$\Rightarrow (A_k)_{ref}\left(\frac{1}{B_{n+1}} - \frac{1}{B_n}\right) = \frac{2\pi q}{\hbar}$$

III. Landau's levels – energy degeneracy

1. Assume the constant B field is along the z-axis. We need only to consider the k_x-k_y plane. The wave function of the electron can be written as

$$\Psi = \varphi(k_x, k_y)e^{ik_z z}$$

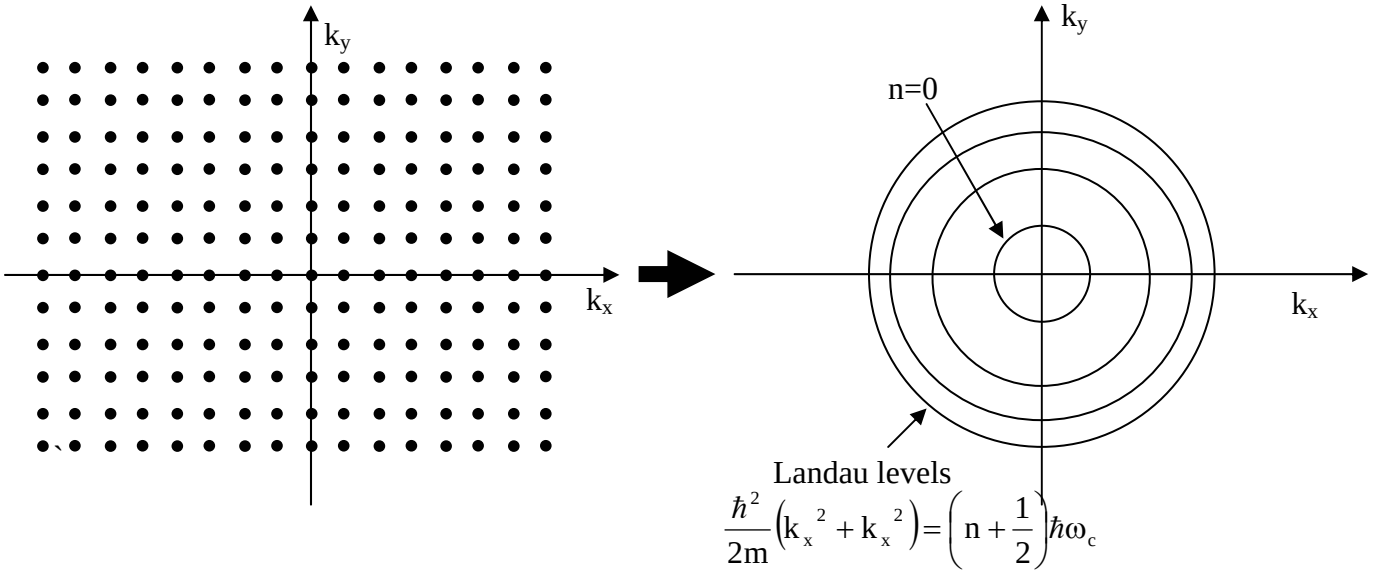
2. The electron will trace equal energy path in the k_x-k_y plane. These paths are quantized so that

$$(A_k)_n = \pi k_n^{\ 2} = \frac{2n\pi qB}{\hbar}$$

$$\text{or } \Delta(A_k) = (A_k)_{n+1} - (A_k)_n = \frac{2\pi qB}{\hbar}$$

Note that ΔS is independent of n.

3. State will be rearrange from the original "square grids" into landau levels:



4. All states between the n-th and (n+1)-th circles will "collapse" into the (n+1)-th circles. This causes the high degeneracy of the Landau levels.

5. Area between the (n-1)-th and n-th circles is

$$\Delta\left(A_k\right) = \left(A_k\right)_{n-1} - \left(A_k\right)_n = \frac{2\pi qB}{\hbar}$$

which is a constant between any two consecutive circles (but proportional to B).
$\therefore$ degeneracy D of the n-th level is

$$\text{degenercy D} = \text{number of states in the n - level} = \frac{\Delta\left(A_k\right)}{\left(\frac{(2\pi)^2}{A}\right)} = \frac{2\pi qB}{\hbar\left(\frac{(2\pi)^2}{A}\right)} = \frac{qA}{2\pi\hbar}B = \rho B$$

Notes.
(i) All Landau level n hold the same number of states (i.e. same degeneracy).
(ii) The degeneracy D is proportional to B. The proportional constant depends only on sample size A.
(iii) If spin is considered, each Landau level will split into two because of the interaction between the spin and the field B.
(iv) It takes a large number of landau levels to hold all the electrons. For example, consider a $1cm^3$ sample in a magnetic field of 1T.

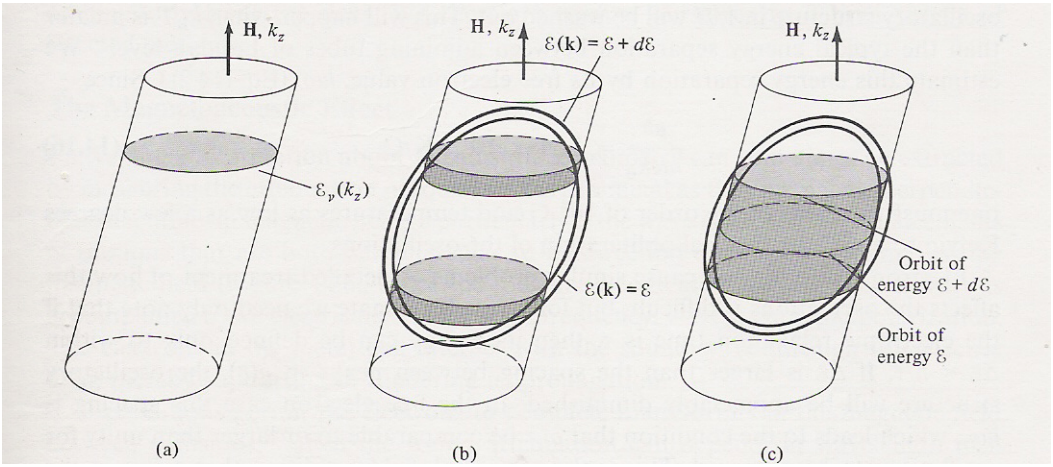
$$D = \frac{qA}{2\pi\hbar} B = \frac{1.6\times10^{-19}\times(10^{-2})^2}{2\pi\times1.05\times10^{-34}} = 2.4\times10^{10}$$

$$\text{No. of electrons} \sim \left(\frac{10^{-2}}{10^{-10}}\right)^2 = 10^{16}$$

$$\therefore n \sim \frac{10^{16}}{2.4\times10^{10}} \sim 4\times10^{5}$$

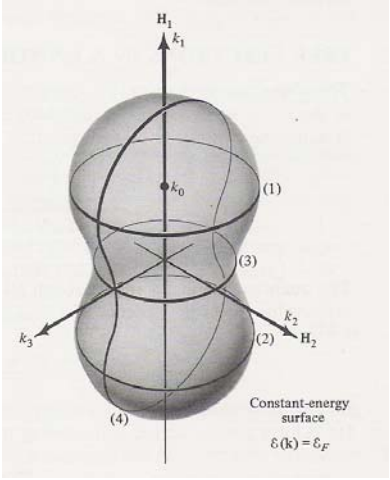
IV. De Hass-van Alphen (DhvA) effect

1. If the Landau level is imposed the Fermi surface, the situation can be described by the following figure:



(a) shows the n-th Landau tube with energy equal to the Fermi energy of the sample. Only electrons at the intersection between this Landau tube and the Fermi surface can remain in a "coherent state", as shown in (b), which is relatively small amount. However, when the Landau tube is "tangent" to the Fermi surface as shown in (c), the number of these "coherent" electrons will increase dramatically and situation like this corresponds to a singularity in the density of states.

2. When the Landau tube is tangent to a Fermi surfaces (as in (c) above), the intersection is a path called *extremal orbit.* Extremal orbit is always perpendicular to the applied field.



3. Since

$$\left(\mathrm{A_k}\right)_{ref}\left(\frac{1}{B_{n+1}}-\frac{1}{B_n}\right)=\frac{2\pi q}{\hbar}$$

Now the reference surface is actually the Fermi surface, i.e $(A_k)_{ref}=(A_k)_{extremal}$ orbit, therefore the Landau tube will touch the extremal orbit when the magnetic field is changed by

$$\left(\mathrm{A_k}\right)_{\text{extremal orbit}}\left(\frac{1}{B_{n+1}}-\frac{1}{B_n}\right)=\frac{2\pi q}{\hbar}\Rightarrow\Delta\left(\frac{1}{\mathrm{B}}\right)=\frac{2\pi q}{\hbar\left(\mathrm{A_k}\right)_{\text{extremal orbit}}}$$

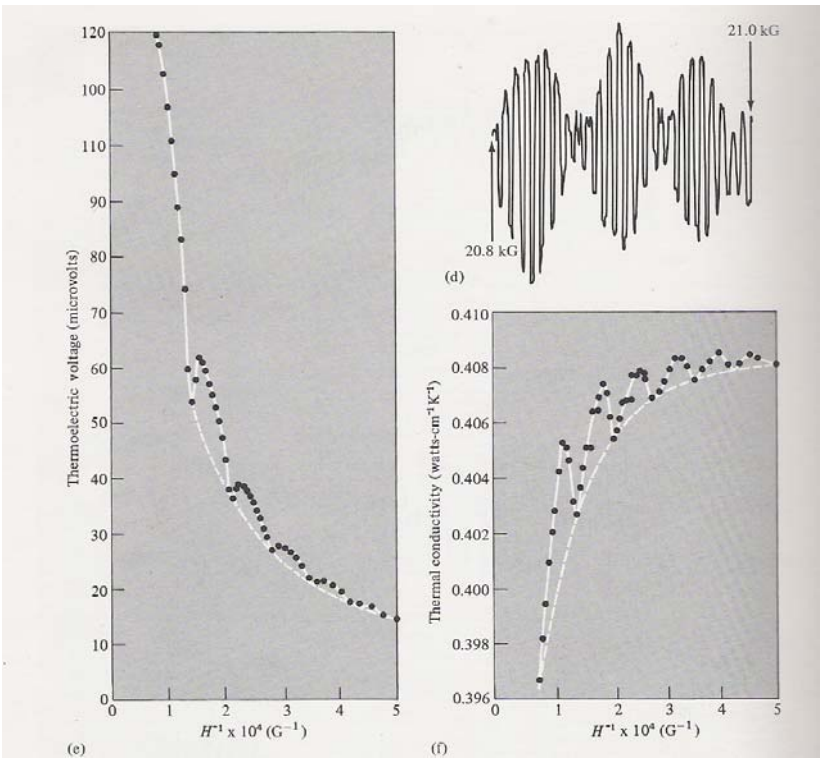
4. Any electronic properties that depend on the density of state at the Fermi level, like magnetization and conductivity, will oscillate with 1/B with a period given by

$$\Delta\left(\frac{1}{\mathrm{B}}\right)=\frac{2\pi q}{\hbar\left(\mathrm{A_k}\right)_{\text{extremal orbit}}}$$

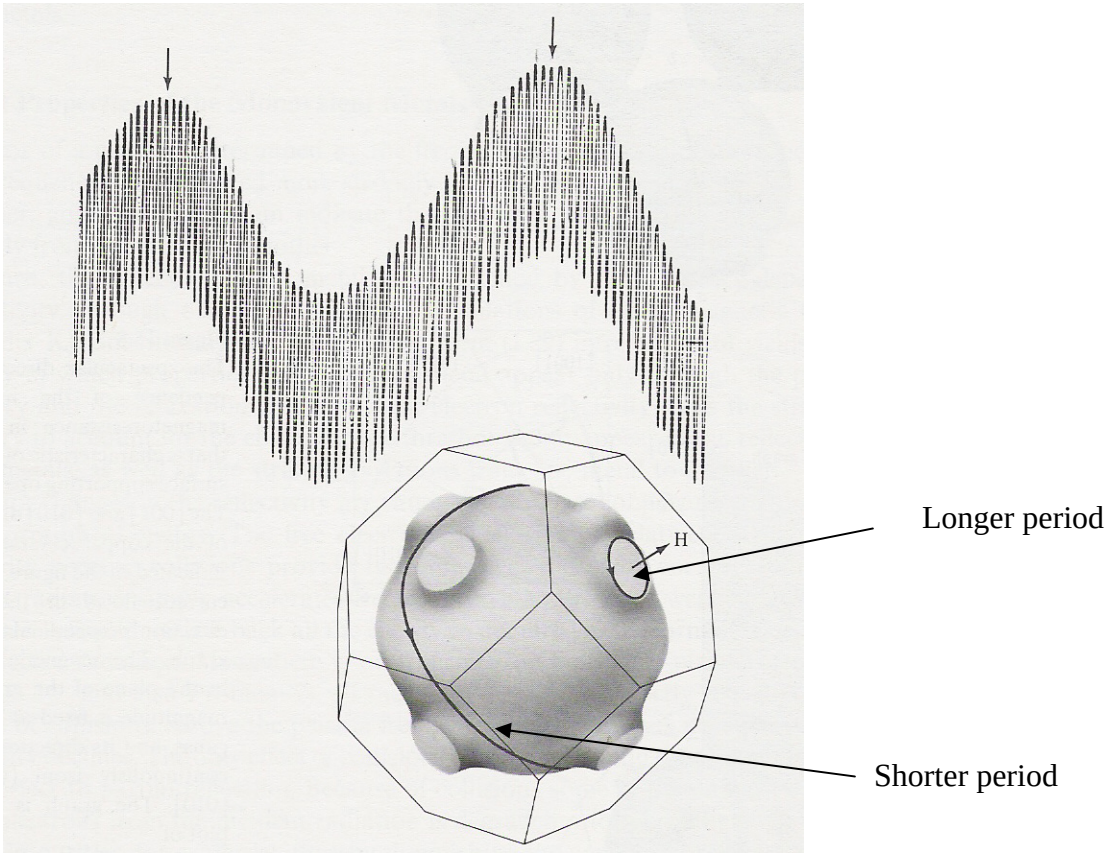
5. By measuring the period of this oscillation, we can derive the area of the extremal orbit.

$$\left(\mathrm{A_k}\right)_{\text{extremal orbit}}=\frac{2\pi q}{\hbar\Delta\left(\frac{1}{\mathrm{B}}\right)}$$

6. Oscillation of magnetization M is known as De Hass-van Alphen (DhvA) effect. Oscillation of conductivity is known as Shubnikov- de Hass effect. Quantities like thermoelectric power and thermal conductivity oscillate too.



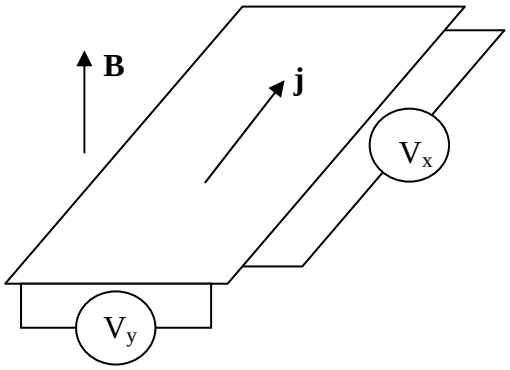
7. Example of de Haas- van Alphan effect:



8. The Fermi surface can be obtained by measuring the de Haas –van Alphen effect at different directions.

V. Quantum Hall effect

1. Quantum Hall effect occurs in 2-D electron gas.



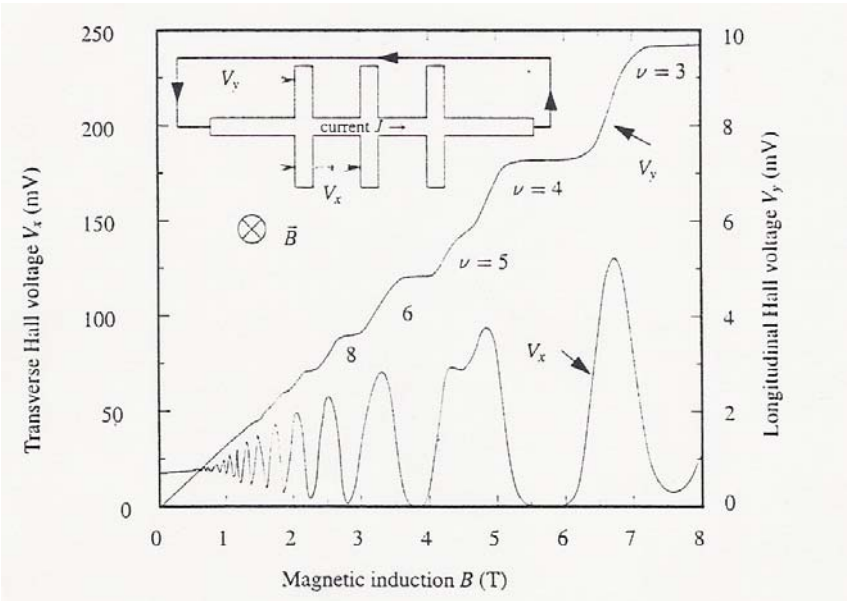
$E_x = \rho_L J_x$

$E_y = \rho_T J_x$

$$\text{Drude model}: \rho_L = \frac{1}{\sigma_L} = \frac{m}{ne^2\tau}$$

$$\text{Hall effect}: \rho_T = -\frac{B}{ne}$$

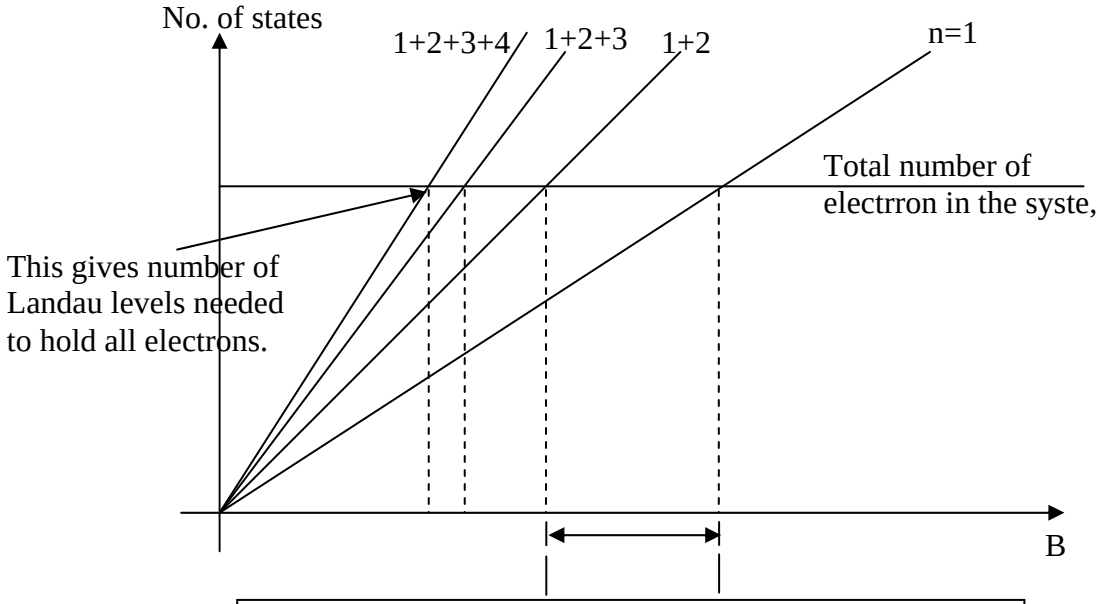
2. Result when the electron gas is two-dimensional:



Experimental observation of the integer quantum Hall effect. A constant current of $J = 28\ \mu$A runs around the upper loop; the current between the terminals separated by V_y is much smaller. The voltage plateaus correspond to Hall resistances of R_H/ν, where ν is an integer. [Source: Cage (1987), p. 44.]

3. As can be seen in above figure, V_y (i.e. ρ_T) forms steps. There are two things to explain in the data: the step width and the step height.

4. On the step width: note that the step width is not uniform, and it is becoming wider and wider as B is increased. The step width is actually uniform with respect to 1/B. The reason for this is the same as that of dHvA effect.



$$N = n\rho B_n = (n+1)\rho B_{n+1} \Rightarrow \frac{N}{B_{n+1}} - \frac{N}{B_n} = \rho$$

$$\Rightarrow \frac{1}{B_{n+1}} - \frac{1}{B_n} = \frac{\rho}{N} = \text{constant}$$

For simplicity, let us assume at a certain field B_n when the Landau levels up to n are full. If B is increased slightly, each Landau level can hold more electrons and as a result, the highest n-level will become only partially full. When B is increased exactly an amount of

$$\Delta\left(\frac{1}{B}\right) = \frac{\rho}{N}$$

the n-th level will become completely empty and the (n-1)-level is now the highest level to be completely full. N is the total number of electrons in the system.

5. On the step height: let the B-field for the midpoint of the i-th step is B_i and assume this corresponds to the case when the i-level is full while all higher levels are empty.

$$\therefore \text{number of conducting electrons per unit area (real space)} = \frac{N}{A} = \frac{Di}{A}$$

$$= \frac{eA}{2\pi\hbar} B_i \cdot \frac{i}{A}$$

$$= \frac{ei}{2\pi\hbar} B_i$$

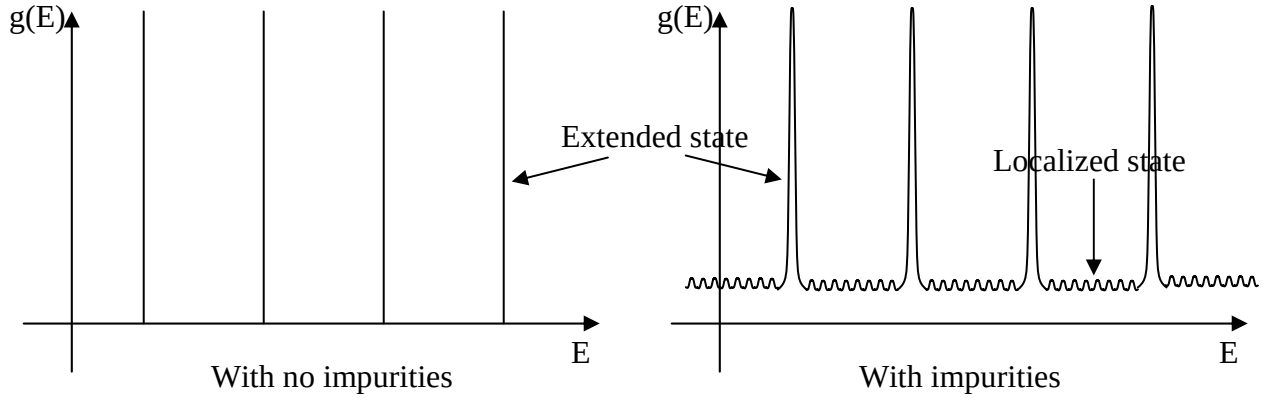
$$\rho_T = -\frac{B_i}{ne} = \frac{B_i}{\left(\frac{ei}{2\pi\hbar} B_i\right)e} = \frac{2\pi\hbar}{ie^2} = \frac{h}{ie^2}$$

or $\quad \sigma_T = \frac{1}{\rho_T} = \frac{ie^2}{h}$

$\therefore$ Hall conductivity is quantized, with 1 quanta $= \frac{e^2}{h}$

6. The above discussions only help to estimate the step width and step height in quantum hall effect, but do not explain the formation of the steps! The problem is from the fact that as B is increased and the Landau tubes "spread outward", i (highest Landau tube with electrons in it) will indeed decrease as the degeneracy of the tubes increase. However, this only transfers electrons from higher tubes to lower tubes and it has *no effect* on n. In other words, this has no effect on ρ_T (=B/ne) and ρ_T will still increase in proportion to B.

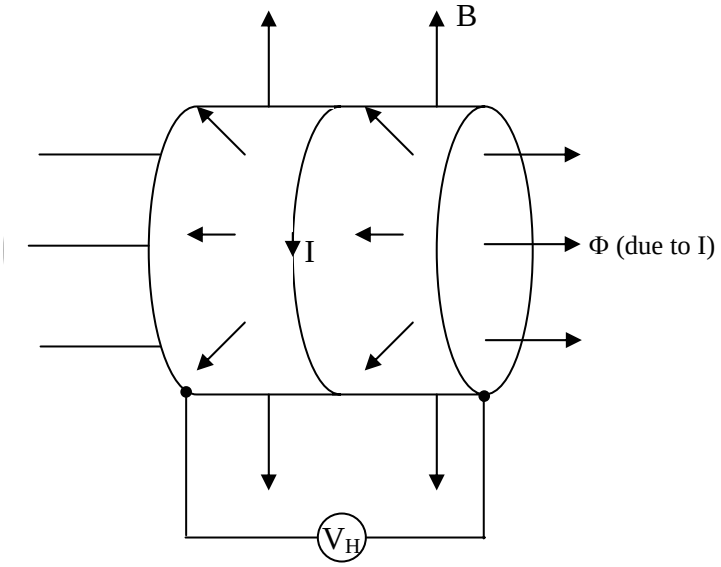
To explain the formation of the transverse resistivity (or conductivity) steps in quantum hall effect, we need impurities in the sample. These impurities provide localized states that play a critical role in quantum hall effect.



These localized (i.e. impurities) serve as an electron reservoir. As a result, transfer of electrons occur between the Landau tube (extended states) and the impurities (localized states), but not between the Landau tubes. For example, as the B field is increased, as the highest level will become partially full, but in reality it will immediately suck up electrons from the localized states and remains full. Hence, *the Landau tubes are either*

full or empty. n=Di where i is the highest full Landau level. D increases proportional to B in the $\Delta(1/B)$ period when i is the same, as a result, ρ_T =B/ne is constant during this period of i. As B is increase, to the degree that the i-level can be emptied, all the electrons in the i-level will suddenly return to the localized state and this will cause a sudden increase in ρ_T because of the reduction in n. This is how the step is formed.

7. Laughlin's thought experiment. The quantization of Hall conductivity can also be derived from the Laughlin's though experiment. This originates from the fact that this is a quantum effect and the result should be independent of the actual geometry of the device. We can imagine the following device:



The 2-D electronm gas is rolled in a cylinder of unit dimension with the B-field attached to it in perpendicular, as shown in the above figure. The circulating current I (J_x in the original set up) produces a flux Φ through the cylinder. We can calculate $R_T = V_H / I$ as follow.
Let U be the total energy of the resistanceless system.

$$\therefore \quad \frac{\partial U}{\partial t} = -IV_x$$

By Faraday's Law:

$$V_x = -\frac{\partial \Phi}{\partial t}$$

$$\therefore \frac{\partial U}{\partial t} = -I\left(-\frac{\partial \Phi}{\partial t}\right) \Rightarrow I = \frac{\partial U}{\partial \Phi}$$

Change in Φ will cause a change in energy U and this will also add electrons to the system (from the localized states). However, if $\delta\Phi=\Phi_0 = h/e$, all extended orbits are identical to those before the flux quanta is added. If i electrons enter the system from the left at the beginning, the same amount of electrons has to be removed from the right when the increment in Φ reaches one flux quanta.

$$\therefore \delta U = ieV_H$$

$$\text{But } I = \frac{\partial U}{\partial \Phi} \Rightarrow \delta U = I\delta\Phi = I\Phi_0 = I\frac{h}{e}$$

$$\therefore ieV_H = I\frac{h}{e} \Rightarrow \sigma_T = \frac{I}{V_H} = i\frac{e^2}{h}$$

8. When the sample is very clean, quantum hall effect will not disappear as expected. Instead, fractional quantum hall effect will occur.

$$\sigma_T = \frac{p}{q}\frac{e^2}{h}$$

Fractional quantum hall effect can be explained by composite particles. These particles are actually electrons bound to magnetic flux quanta.

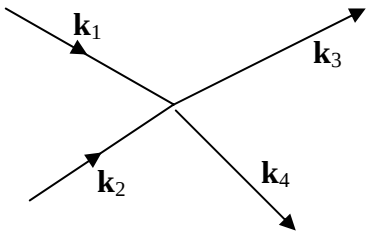
VI. Electron-electron interactions

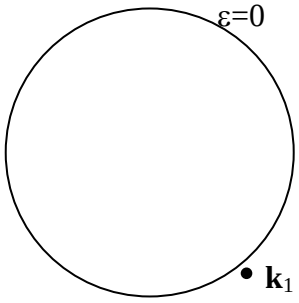
1. A Fermi gas is a system of non-interacting fermions (like electrons); the same system with interactions is a Fermi liquid. We can imagine there is a knob to turn the interactions gradually on and off.

2. Landau's theory on Fermi liquid: Excitation of a Fermi liquid is one-one correspondence to a single particle excitation of the Fermi gas. We can follow this one-one correspondence by turn the knob of interactions. Therefore, excitation of a Fermi liquid can be consider as a particle, called quasiparticle. It may be thought of a single particle dressed with a distortion cloud ub the electron gas. Since the two systems are not exactly the same, quasiparticle has lifetime.

3. Landau's theory simplifies the Fermi liquid dramatically. Its success lies in the fact that electron-electron interaction has a long mean free path. Reasons why it has a long mean free path:

(i) Pauli's exclusion principle: In a collision, both energy and momentum have to be conserved.





Let $\mathbf{k}_1$ be an electron just outside the Fermi sphere as shown in the above figure, with an energy $\varepsilon_1>0$ (let the Fermi energy be $\varepsilon=0$). It is likely to collide with another electron $\mathbf{k}_2$,

inside the Fermi sphere $\varepsilon_2<0$. Pauli's principle requires $\mathbf{k}_3$ and $\mathbf{k}_4$ to be outside the sphere, because all states inside the sphere are already occupied:

$$\underbrace{\varepsilon_1}_{>0} + \underbrace{\varepsilon_2}_{<0} = \underbrace{\varepsilon_3 + \varepsilon_4}_{>0}$$

This means $\mathbf{k}_2$ has to lie within a thin shell of ε_1 below the Fermi surface:

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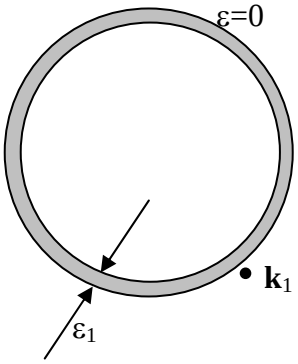
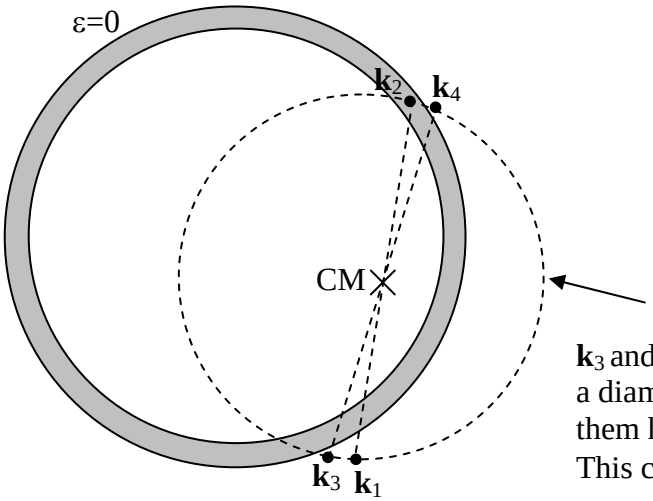


By consideration of conservation of energy only, a factor of

$$\frac{4\pi k_F^{\,2}\delta k}{\frac{4}{3}\pi k_F^{\,3}} \approx \frac{\varepsilon_1}{\varepsilon_F}$$

Electrons below the Fermi surface can collide with $\mathbf{k}_1$.
By considering conservation of momentum, we know that not all $\mathbf{k}_3$ and $\mathbf{k}_4$ are possible:



$\mathbf{k}_3$ and $\mathbf{k}_4$ have to stay at the two ends of a diameter of this sphere and both of them have to remain outside the sphere. This causes another reduction of $(\varepsilon_1/\varepsilon_F)$.

Hence the cross section of electron-electron interaction is reduce by a total factor of

$$\left(\frac{\varepsilon_1}{\varepsilon_F}\right)^2 \approx \left(\frac{k_B T}{\varepsilon_F}\right)^2$$

For $\varepsilon_F \sim 10^5$K, this corresponds to a factor of 10^4.

(ii) Screening effect:

$$\phi(\vec{r}) = \frac{Q_{ext}}{r} e^{-k_s r}$$

The electric potential due to an external charge Q_{ext} placed in the electron sea decay exponential with a length scale called Thormas Fermi screening length ($1/k_s$):

$$k_s = \left[\frac{6\pi n_0 e^2}{E_F} \right]^{1/2}$$