University of Kentucky
Department of Physics and Astronomy

PHY 525: Solid State Physics II

Fall 2000

Final Examination

Solutions

Date: December 11, 2000 (Monday)

Time allowed: 120 minutes.

Answer all questions.

1. *Heat capacity of ferromagnets.* (a) Use the approximation magnon dispersion relation $\omega=Ak^2$ to find the leading term in the heat capacity of a three-dimensional ferromagnet at low temperatures $k_BT<<J$. The result is $0.113k_B(k_BT/\leftarrow A)3/2$, per unit
volume. The definite intergral $\int_0^\infty \frac{x^{\frac{3}{2}}dx}{[\exp(x)-1]}$ is given as $\frac{3}{4}\sqrt{\pi}\,\zeta\left(\frac{5}{2}\right)$ with $\zeta\left(\frac{5}{2}\right)\approx 1.3415$
approximately. (b) How does the heat capacity of a metal depend on temperature at low T (just write down the result, you do not need to do any derivation)? Free electron has a similar dispersion relation as E ;k^2, why the heat capacities of metal and ferromagnet depend differently on temperature at low T?

Solution.

$$U=\int_0^\infty \frac{D(\omega)\hbar\omega d\omega}{\exp\left(\frac{\hbar\omega}{k_BT}\right)-1} \qquad ----(1)$$

To calculate $D(\omega)$:

$$D(\omega)d\omega=\frac{4\pi k^2dk}{V_k} \quad\Rightarrow\quad D(\omega)d\omega=\frac{4\pi k^2dk}{\frac{8\pi^3}{V}}$$

$$\Rightarrow\quad D(\omega)=\frac{4\pi k^2}{8\pi^3}\frac{dk}{d\omega}$$

$$\Rightarrow\quad D(\omega)=\frac{k^2}{2\pi^2}\frac{dk}{d\omega}$$

For $\omega=Ak^2, \frac{dk}{d\omega}=2Ak=2A\sqrt{\frac{\omega}{A}}=2\sqrt{A\omega}$

$$\therefore\ D(\omega)=\frac{\omega}{2\pi^2A}\frac{1}{2\sqrt{A\omega}}=\frac{1}{4\pi^2}\omega^{\frac{1}{2}}A^{-\frac{3}{2}} \qquad ---(2)$$

Substitute this into (1) :

$$U = \int_0^\infty \frac{\hbar\omega^{\frac{3}{2}} d\omega}{4\pi^2 A^{\frac{3}{2}}\left[\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1\right]}$$

$$\text{Let } \frac{\hbar\omega}{k_B T} = x, \ \omega = \frac{k_B T}{\hbar} x$$

$$U = \frac{\hbar\left(\frac{k_B T}{\hbar}\right)^{\frac{5}{2}}}{4\pi^2 A^{\frac{3}{2}}} \int_0^\infty \frac{x^{\frac{3}{2}} dx}{[\exp(x) - 1]}$$

Given::

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> int (x^(3/2)/(exp(x)-1),x=0..infinity);
>
>
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$$\frac{3}{4}\sqrt{\pi}\,\zeta\left(\frac{5}{2}\right)$$

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>
> Zeta(2.5);
>
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1.341487257

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>
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$$\therefore \quad U = \frac{\hbar\left(\frac{k_B T}{\hbar}\right)^{\frac{5}{2}}}{4\pi^2 A^{\frac{3}{2}}} \frac{3}{4}\sqrt{\pi}\zeta\left(\frac{5}{2}\right) = \frac{1}{3}\left(\frac{1}{\hbar\pi A}\right)^{\frac{3}{2}} (k_B T)^{\frac{5}{2}} \zeta\left(\frac{5}{2}\right)$$

$$C = \frac{\partial U}{\partial T} = \frac{3}{16} \cdot \frac{5}{2}\left(\frac{1}{\hbar\pi A}\right)^{\frac{3}{2}} (k_B T)^{\frac{3}{2}} \zeta\left(\frac{5}{2}\right)$$

$$= \underline{\underline{\frac{15}{32}\left(\frac{k_B T}{\hbar\pi A}\right)^{\frac{3}{2}} \zeta\left(\frac{5}{2}\right)}}$$

$$\text{Numerically,} \quad C = \left[\frac{15}{32}\zeta\left(\frac{5}{2}\right)\frac{1}{\pi\sqrt{\pi}}\right]\left(\frac{k_B T}{\hbar A}\right)^{\frac{3}{2}}$$

$$= \left[\frac{15}{32} \times 1.3415 \times \frac{1}{\pi\sqrt{\pi}}\right]\left(\frac{k_B T}{\hbar A}\right)^{\frac{3}{2}}$$

$$= \underline{\underline{0.113\left(\frac{k_B T}{\hbar A}\right)^{\frac{3}{2}}}}$$

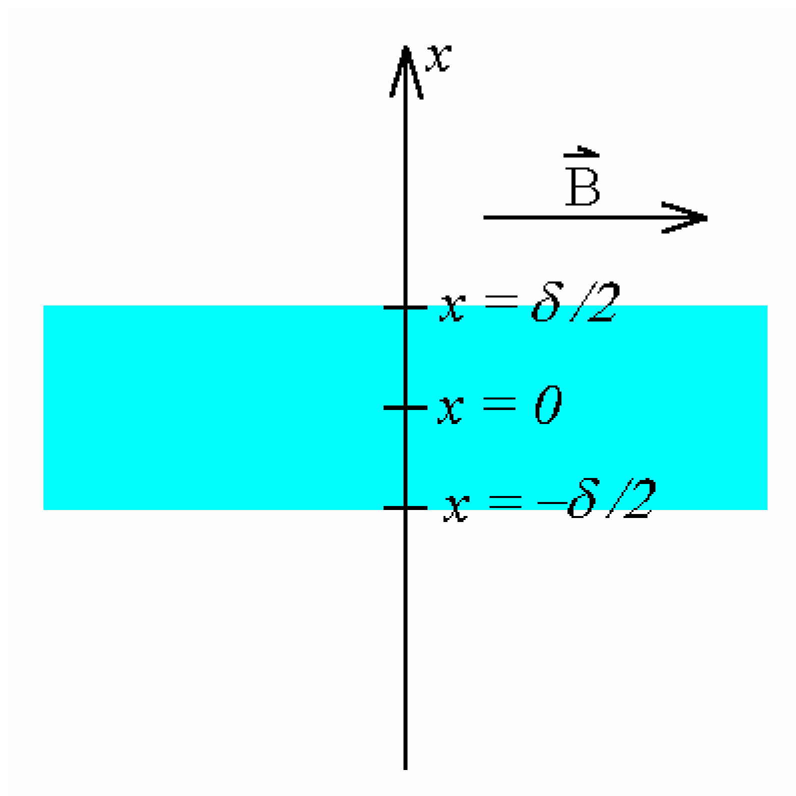
2. *Magnetic field penetration in a plate.* The penetration equation may be written as $\lambda^2\nabla^2 B = B$, where λ is the penetration depth. (a) Show that B(x) inside a superconding plate perpendicular to the x axis and of thickness δ is given by

$$B(x) = B_a \frac{\cosh(x/\lambda)}{\cosh(\delta/2\lambda)},$$

where B_a is the field outside the plate and parallel to it; here x=0 is at the center of the plate. (b) The effective magnetization M(x) in the plane is defined by $B(x)-B_a = 4\pi M(x)$. Show that, in CGS, $4\pi M(x) = -B_a(1/8\lambda^2)(\delta^2 - 4x^2)$, for $\delta \ll \lambda$. In SI, we replace the 4π by μ_0.

Solution.

(a)



$$\lambda^2\nabla^2 B = B \Rightarrow \frac{\partial^2 B}{\partial x^2} - \frac{1}{\lambda^2}B = 0$$

$$\Rightarrow B = Ae^{x/\lambda} + Ce^{-x/\lambda}$$

At $x = \delta/2$ and $x = -\delta/2$, $B = B_a$

$$\therefore Ae^{\delta/2\lambda} + Ce^{-\delta/2\lambda} = B_a \quad ---(1)$$

$$Ae^{-\delta/2\lambda} + Ce^{\delta/2\lambda} = B_a \quad ---(2)$$

$(1) \Rightarrow A = B_a e^{-\delta/2\lambda} - Ce^{-\delta/\lambda}$

Substitute this into (2),

$$(B_a e^{-\delta/2\lambda} - Ce^{-\delta/\lambda})e^{-\delta/2\lambda} + Ce^{\delta/2\lambda} = B_a$$

$$\Rightarrow C(e^{\delta/2\lambda} - e^{3\delta/2\lambda}) = B_a(1 - e^{-\delta/\lambda})$$

$$\Rightarrow C(1 - e^{-\delta/\lambda})(e^{\delta/2\lambda} + e^{-\delta/2\lambda}) = B_a(1 - e^{-\delta/\lambda})$$

$$\Rightarrow C = \frac{B_a}{e^{\delta/2\lambda} + e^{-\delta/2\lambda}} = \frac{2B_a}{\cosh\frac{\delta}{2\lambda}}$$

$$\therefore A = B_a e^{-\delta/2\lambda} - Ce^{-\delta/\lambda} = B_a e^{-\delta/2\lambda} - \frac{e^{-\delta/\lambda}B_a}{e^{\delta/2\lambda} + e^{-\delta/2\lambda}}$$

$$= B_a \frac{1 + e^{-\delta/\lambda} - e^{-\delta/\lambda}}{e^{\delta/2\lambda} + e^{-\delta/2\lambda}}$$

$$= \frac{B_a}{e^{\delta/2\lambda} + e^{-\delta/2\lambda}}$$

$$= \frac{B_a}{2\cosh\frac{\delta}{2\lambda}}$$

$$\therefore B(x) = Ae^{x/\lambda} + Ce^{-x/\lambda} = \frac{B_a}{2\cosh\frac{\delta}{2\lambda}}e^{x/\lambda} + \frac{B_a}{2\cosh\frac{\delta}{2\lambda}}e^{-x/\lambda}$$

$$= \frac{B_a}{2\cosh\frac{\delta}{2\lambda}}(e^{x/\lambda} + e^{-x/\lambda})$$

$$= B_a \frac{\cosh\frac{x}{\lambda}}{\cosh\frac{\delta}{2\lambda}}$$

(b)

$$4\pi M(x) = B(x) - B_a = B_a \frac{\cosh\frac{x}{\lambda}}{\cosh\frac{\delta}{2\lambda}} - B_a$$

$$= B_a\left(\frac{\cosh\frac{x}{\lambda}}{\cosh\frac{\delta}{2\lambda}} - 1\right)$$

$$\approx B_a\left(\frac{1 + \frac{1}{2}\left(\frac{x}{\lambda}\right)^2}{1 + \frac{1}{2}\left(\frac{\delta}{2\lambda}\right)^2} - 1\right) \qquad (\delta << \lambda)$$

$$= B_a\left(\frac{1 + \frac{1}{2}\left(\frac{x}{\lambda}\right)^2 - \left[1 + \frac{1}{2}\left(\frac{\delta}{2\lambda}\right)^2\right]}{1 + \left(\frac{\delta}{2\lambda}\right)^2}\right)$$

$$\approx B_a\left[\frac{1}{2}\left(\frac{x}{\lambda}\right)^2 - \frac{1}{2}\left(\frac{\delta}{2\lambda}\right)^2\right]$$

$$= -\frac{B_a}{8\lambda^2}\left(\delta^2 - 4x^2\right)$$

3. *Susceptibility of d- and f- electrons.* (a) Apply Hund rule to determine the ground state of (i) Dy^{3+} in the configuration of $4f^9 5s^2 p^6$, and (ii) Cu^{2+} in the configuration of $3d^9$. Express your answer in standard atomic notation. (b) The Curie constant C is given as

$$C = \frac{Ng^2 [J\{J+1)]^2 \mu_B{}^2}{3k_B}$$

for f-electrons. Estimate the Curie constants for Dy^{3+} and Cu^{2+}. Avogadro's number is given as 6.022×10^{23} mol^{-1}. Note that the g-factor is given by the Landé equation:

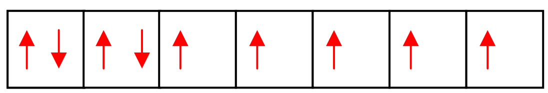
$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$$

(c) What is the ratio $\chi(300K)/\chi(4.2K)$ of the same paramagnetic material, for small field?

<u>Solution.</u>

(a)

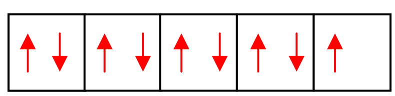
Dy^{3+}:



L = +3 +2 +1 0 -1 -2 -3

$\Sigma L = 5$ (i.e. H)
$\Sigma S = 5/2$, $2S+1 = 6$
$\Sigma J = L+S = 15/2$
$\therefore$ The ground state is $^6H_{15/2}$.

Cu^{2+}:



L = +2 +1 0 -1 -2

$\Sigma L = 2$ (i.e. D)
$\Sigma S = 1/2$, $2S+1 = 2$
$\Sigma J = L+S = 5/2$
$\therefore$ The ground state is $^2D_{5/2}$.

(b) $C = \frac{Np^2 {\mu_B}^2}{3k_B}$, $p = g\sqrt{j(j+1)}$

(i) For Dy^{3+}, $J = 15/2$, $S = 5/2$, $L = 5$

$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} = 1 + \frac{7.5\times 8.5 + 2.5\times 3.5 - 5\times 6}{2\times 7.5\times 8.5} = 1.333$$

$$\therefore p = 1.333\times\sqrt{7.5\times 8.5} = 10.65$$

$$C = \frac{Np^2 {\mu_B}^2}{3k_B} = \frac{6.022\times 10^{23}\times 10.65^2\times (9.274\times 10^{-21})^2}{3\times 1.38\times 10^{-16}}$$

$$= \underline{\underline{14.19\ \ cm^3 K mol^{-1}}}$$

(ii) For Cu^{2+}, $J = 5/2$, $S = 1/2$, $L = 2$

$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} = 1 + \frac{2.5\times 3.5 + 0.5\times 1.5 - 2\times 3}{2\times 2.5\times 3.5} = 1.2$$

Since there is orbital angular momentum quenching in d - electrons, it is better to replace the J by S and the g - factor by 2 in calculating p.

$$\therefore p = 2\times\sqrt{0.5\times 1.5} = 1.73$$

$$C = \frac{Np^2 {\mu_B}^2}{3k_B} = \frac{6.022\times 10^{23}\times 1.73^2\times (9.274\times 10^{-21})^2}{3\times 1.38\times 10^{-16}}$$

$$= \underline{\underline{0.37\ \ cm^3 K mol^{-1}}}$$

(c) For paramagnetism, $\chi = \frac{C}{T}$

$\therefore$ For the same type of ion, $\frac{\chi(300K)}{\chi(4.2K)} = \frac{4.2}{300} = \underline{\underline{1.4\times 10^{-2}}}$

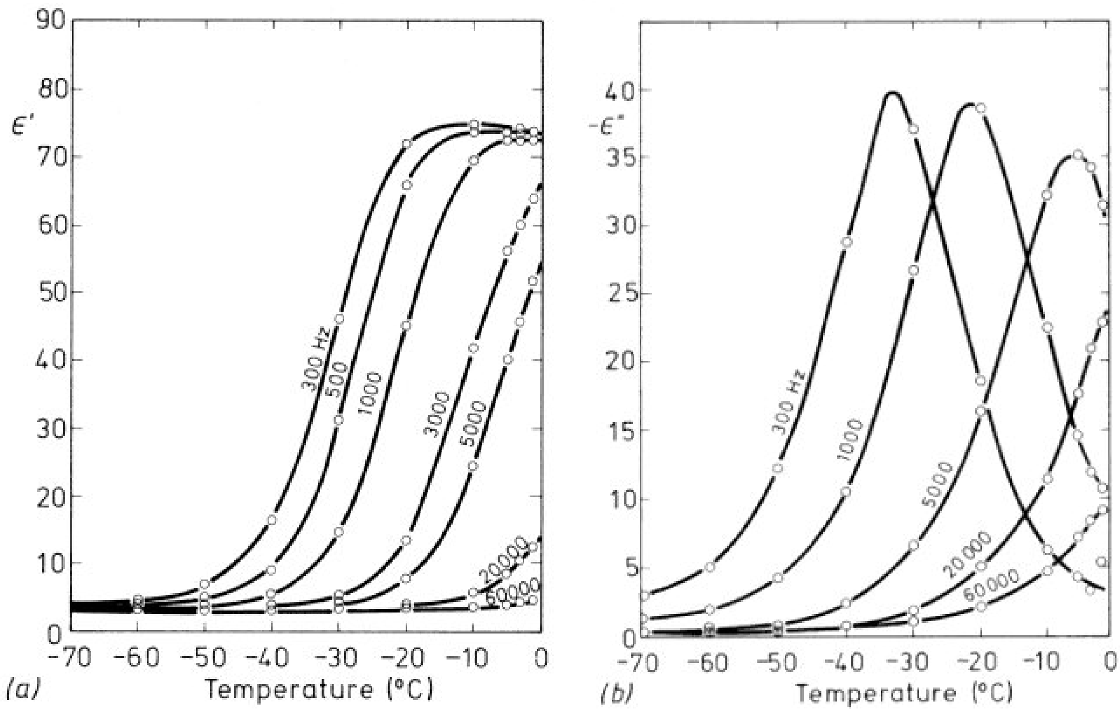
4. *Molecular rotation in solid.* We can model the polarization vector **P** always relaxes towards an equilibrium value $\mathbf{P}_E$ at a rate called relaxation time τ. $\mathbf{P}_E$ depends on the electric field **E** as . The relaxation equation can then be written as

$\frac{d\mathbf{P}}{dt} = \frac{\mathbf{P}_E - \mathbf{P}}{\tau}$. (a) By considering $\mathbf{E} = \mathbf{E}_0 e^{i\omega t}$, show that

$$\mathbf{P}_E = \chi(0)\mathbf{E}$$

$$\varepsilon'(\omega) = 1 + \frac{4\pi\chi(0)}{1+\omega^2\tau^2} \text{ and } \varepsilon''(\omega) = -\frac{i4\pi\omega\chi(0)}{1+\omega^2\tau^2}$$

(b) At what ω is the dielectric loss maximum? What are the relaxation times at frequencies 300Hz, 1000Hz, and 5000Hz? (c) Real and imaginary parts of the dielectric constant of ice as a function of temperature at different frequencies are given in the figure. Show that the temperature dependence is consistent with $\tau = \tau_0 \exp(T_0 / T)$ and find values of τ_0 and T_0. (d) What is the physical basis for a temperature dependence of the kind given in part (c)?



Solution.

(a)

With $E = E_0 e^{i\omega i}$,

$P = \chi(\omega)E \Rightarrow P = \chi(\omega)E_0 e^{i\omega i}$

$P_E = \chi(0)E \Rightarrow P = \chi(0)E_0 e^{i\omega i}$

Substitute these into the relaxation equation,

$$\therefore i\omega(\omega)\chi E_0 = \frac{[\chi(0) - \chi(\omega)]E_0}{\tau} \Rightarrow (1 + i\omega\tau)\chi(\omega) = \chi(0)$$

$$\Rightarrow \chi(\omega) = \frac{\chi(0)}{(1 + i\omega\tau)} \quad \text{---}(1)$$

$$E = D - 4\pi P \quad \Rightarrow E = \varepsilon E - 4\pi\chi E$$

$$\Rightarrow \varepsilon = 1 + 4\pi\chi \quad \text{---}(2)$$

Substitute this into (1),

$$\varepsilon(\omega) = 1 + \frac{4\pi\chi(0)}{(1 + i\omega\tau)}$$

$$= 1 + \frac{4\pi\chi(0)[1 - i\omega\tau]}{1 + \omega^2\tau^2}$$

$$= \left[1 + \frac{4\pi\chi(0)}{1 + \omega^2\tau^2}\right] - \frac{i\omega 4\pi\chi(0)\tau}{1 + \omega^2\tau^2}$$

$$\therefore \varepsilon'(\omega) = \underline{\underline{1 + \frac{4\pi\chi(0)}{1 + \omega^2\tau^2}}} \quad \text{and} \quad \varepsilon''(\omega) = \underline{\underline{-\frac{\omega 4\pi\chi(0)\tau}{1 + \omega^2\tau^2}}}$$

(b)

Dielectric loss is maximum if ε'' is maximum,

$$\frac{d\varepsilon''}{d\omega} = 0 \Rightarrow \frac{d}{d\omega}\left[\frac{4\pi\omega\chi(0)\tau}{1 + \omega^2\tau^2}\right] = 0$$

$$\Rightarrow (1 + \omega^2\tau^2) - \omega(2\omega\tau^2) = 0$$

$$\Rightarrow 1 - \omega^2\tau^2 = 0$$

$$\Rightarrow \underline{\underline{\omega\tau = 1}}, \text{ or } \tau = \frac{1}{2\pi f}$$

For $f = 300\text{Hz}$, $\underline{\underline{\tau = 530.5\mu s}}$

For $f = 1000\text{Hz}$, $\tau = \underline{\underline{159.2\ \mu s}}$

For $f = 5000\text{Hz}$, $\tau = \underline{\underline{31.83\ \mu s}}$

(c)

Since there are only three curves showing a peak in the given data of ε'', and there are two undetermined parameters T_0 and τ_0. To demonstrate that the data is consistent with the given equation, we can use the 300Hz and 1000Hz curves to determine the unknow parameters first, and then check whether these parameters have a reasonable agreement with the 5000Hz curve.

For the 300 Hz curve, the peak occurs at -33^{o}C or 240K :

$$530.5 = \tau_0 \exp\left[\frac{T_0}{240}\right] \Rightarrow 6.274 = \ell n(\tau_0) + \frac{T_0}{240}$$

$$\Rightarrow \; 6.274 = \ell n(\tau_0) + 4.167 \times 10^{-3} T_0 \qquad \text{----(3)}$$

For the 1000 Hz curve, the peak occurs at -22^{o}C or 251K :

$$159.2 = \tau_0 \exp\left[\frac{T_0}{251}\right] \Rightarrow 5.070 = \ell n(\tau_0) + \frac{T_0}{251}$$

$$\Rightarrow \; 5.070 = \ell n(\tau_0) + 3.984 \times 10^{-3} T_0 \qquad \text{----(4)}$$

$$(3) - (4) \;\; \Rightarrow 1.204 = 1.830 \times 10^{-4} T_0$$

$$\Rightarrow \; T_0 = \underline{\underline{6579\text{K}}}$$

Substitute this pack into (3), $6.274 = \ell n(\tau_0) + 4.167 \times 10^{-3} \times 6579$

$$\Rightarrow \ell n(\tau_0) = -21.14$$

$$\Rightarrow \tau_0 = 6.58 \times 10^{-10} \;\mu\text{s} = \underline{\underline{6.58 \times 10^{-16}\ \text{s}}}$$

For the 1000 Hz curve, the peak occurs at -6^{o}C or 267K :

Above parameters require

$$\tau = 6.58 \times 10^{-16} \exp\left[\frac{6579}{267}\right]$$

$$\Rightarrow \qquad \tau = 6.58 \times 10^{-16} \times 5.026 \times 10^{10}$$

$\Rightarrow \qquad \tau = 33.1\,\mu\text{s}$, this is close to the result in part (b).

(d) Relaxing time increases as temperature is lowered. This means a longer time for the rotation to reach equilibrium at low temperature. Together with the exponential relation, we know that there is a barrier defined by T_0 for the rotational motion. Only when the thermal energy $k_B T$ is higher than the barrier, the molecules can rotate fast enough to reduce the relaxation time exponentially.