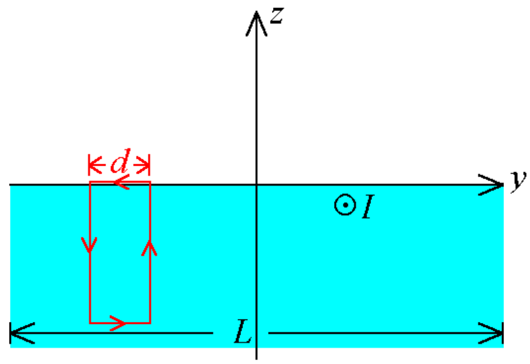
1. ***Current distribution in a superconducting slab*** Consider a semi-infinite superconductor filling the space $z<0$. For simplicity, let us assume the sample has a length of L in the y direction and $L \to \infty$ as shown in the figure. The penetration depth of the superconductor is λ. The superconductor is carrying a current I flowing in the +x direction. (a) What is the magnetic field just outside the superconductor? (b) Use your result in (a) as boundary condition, find the magnetic field inside the superconductor by solving London equation. (c) Determine the current distribution as a function of z (i.e. $J(z)\hat{i}$).



Solution:

(a)
Construct a loop as shown in the diagram.
The magnetic field is in the $-y$ direction, and $\oint \mathbf{B}\cdot d\vec{\ell} = Bd$
Current enclosed by the loop $= I_{enclosed} = Id/L$

$$\text{Ampere's Law} \Rightarrow \oint \mathbf{B}\cdot d\vec{\ell} = \frac{4\pi}{c} I_{enclosed}$$

$$\Rightarrow Bd = \frac{4\pi}{c}\cdot\frac{Id}{L}$$

$$\Rightarrow B = \underline{\underline{\frac{4\pi I}{cL}}} \quad \text{just above the surface of the superconductor.}$$

(b) London equation:

$$\lambda^2 \nabla^2 B = B \Rightarrow \frac{\partial^2 B}{\partial z^2} - \frac{1}{\lambda^2} B = 0$$

$$\Rightarrow B = Ae^{z/\lambda} + Ce^{-z/\lambda}$$

$B \to 0$ as $z \to -\infty$, $\therefore C = 0$.

$B(z) = Ae^{z/\lambda}$.

At $z = 0$, $B(0) = \dfrac{4\pi I}{cL} \Rightarrow A = \dfrac{4\pi I}{cL}$

$B(z) = \dfrac{4\pi I}{cL} e^{z/\lambda}$.

Including direction of the magnetic field,

$$\underline{\underline{\vec{B}(z) = -\frac{4\pi I}{cL} e^{z/\lambda}\hat{j}}}$$

(c) Maxwell equation:

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} \quad \Rightarrow \quad \vec{j} = \frac{c}{4\pi}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -\frac{4\pi I}{cL}e^{z/\lambda} & 0 \end{vmatrix}$$

$$= \frac{c}{4\pi}\frac{\partial}{\partial z}\left[\frac{4\pi I}{cL}e^{z/\lambda}\right]\hat{i}$$

$$= \underline{\underline{\frac{I}{L\lambda}e^{z/\lambda}\hat{i}}}$$

Note that $\int_{-L/2}^{L/2} dy \int_{-\infty}^{0} j(z)dz = \int_{-L/2}^{L/2} dy \int_{-\infty}^{0} \frac{I}{L\lambda}e^{z/\lambda}dz = I,$ as expected.