2. ***Conductivity for free electrons at high frequency.*** (a) Conductivity is defined as $\sigma = j/E$. Since j and E are not necessary in phase, σ can be complex. Let the conductivity of a metal be $\sigma(\omega) = \sigma'(\omega) + i\sigma''(\omega)$, where $\sigma'(\omega)$ and $\sigma''(\omega)$ are the real part and imaginary part of the function respectively. Use Kramers-Kronig relation, show that at high frequency $\omega \to \infty$, $\omega\sigma''(\omega) = \frac{2}{\pi}\int_0^\infty \sigma'(\omega)d\omega$. (b) At very high frequency, the electrons in the metal are essentially oscillated by the electric field without any drifting. Write down the equation of motion of an electron and then show that $\omega\sigma''(\omega) = ne^2/m$.

(c) Further prove that $\int_0^\infty \sigma'(\omega)d\omega = \pi ne^2/2m$. (d) The electron concentration in a copper sample is 8×10^{22} cm^{-3}, mean free path is ~400Å, and the Fermi velocity is 1.6×10^{8} cm s^{-1}. Give a measure on the meaning of "high frequency" for the above results to be valid.

Hint. In case if you forget Kramer-Kronig relation, you can derive it by calculating the integral $P\int_{-\infty}^{+\infty} \frac{\sigma(s)}{s-\omega} d\omega$, and compare real and imaginary parts. Assume $\sigma(\omega)$ has no pole in the upper half of the complex plane, and $\sigma'(\omega)$ is even and $\sigma''(\omega)$ is odd.

Solution:

(a) Kramers - Kronig relation,

$$\sigma''(\omega) = -\frac{2\omega}{\pi} P\int_0^\infty \frac{\sigma'(s)}{s^2-\omega^2} ds$$

For large ω, as long as ω >> upper limit of integration,

$$\sigma''(\omega) \approx -\frac{2\omega}{\pi} P\int_0^\infty \frac{\sigma'(s)}{-\omega^2} ds$$

$$\Rightarrow \quad \sigma''(\omega) \approx \frac{2\omega}{\pi\omega^2} P\int_0^\infty \sigma'(s)\, ds$$

$$\Rightarrow \quad \sigma''(\omega) \approx \underline{\underline{\frac{2}{\pi\omega} P\int_0^\infty \sigma'(s)\, ds}}$$

(b) Equation of motion of an electron:

$$m\frac{dv}{dt} = -eE_0 e^{-i\omega t}$$

$$x = x_0 e^{-i\omega t} \Rightarrow v = \frac{dx}{dt} = -i\omega x_0 e^{-i\omega t}$$

$$= v_0 e^{-i\omega t} \qquad (v_0 = -i\omega x_0)$$

$$m\frac{dv}{dt} = -i\omega v_0 e^{-i\omega t}$$

$$\Rightarrow -im\omega v_0 e^{-i\omega t} = -eE_0 e^{-i\omega t}$$

$$\Rightarrow im\omega v_0 = eE_0$$

$$\Rightarrow v_0 = -\frac{ieE_0}{m\omega}$$

$$\therefore j = n(-e)v_0 = \frac{ne^2E}{m\omega} i$$

$$\therefore \sigma''(\omega) = \frac{ne^2}{m\omega} \Rightarrow \omega\sigma''(\omega) = \frac{ne^2}{m}$$

(c) (a) $\Rightarrow \omega\sigma''(\omega) = \frac{2}{\pi}\int_0^\infty \sigma'(s)\,ds$

$$\therefore \frac{2}{\pi}\int_0^\infty \sigma'(s)\,ds = \frac{ne^2}{m}$$

$$\Rightarrow \int_0^\infty \sigma'(s)\,ds = \frac{\pi ne^2}{2m}$$

(d) The frequency should be fast so that the electrons do not make a lot of collisions during oscillation, i.e. $\omega >> \frac{1}{\tau}$.

$$\omega >> \frac{1}{\tau} = \frac{1}{\frac{\ell}{v_s}} = \frac{v_s}{\ell} \Rightarrow \omega >> \frac{1.6\times10^8}{400\times10^{-8}} = \underline{\underline{4\times10^{13}\,/\,s}}$$