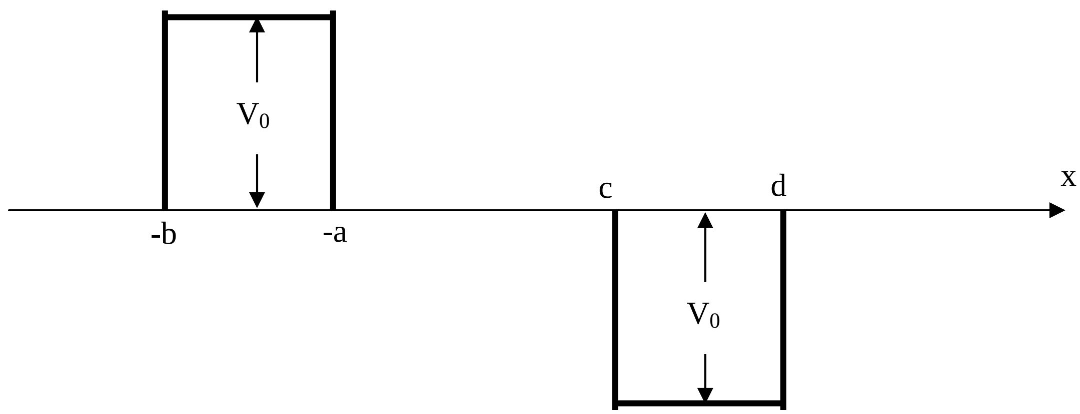
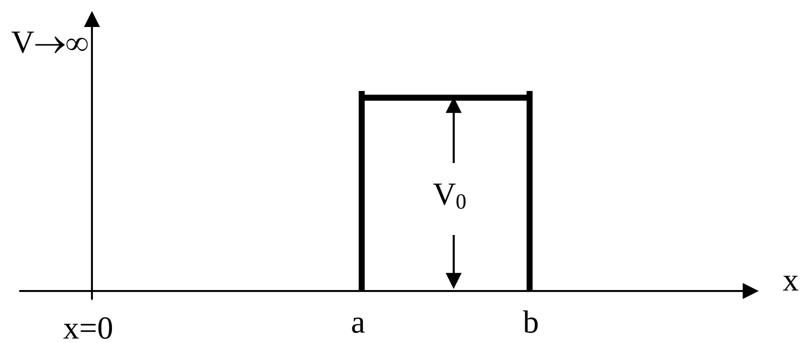
Gasiorowicz 3rd edition Feb 26, 2008

P4-4.

Without actually solving the Schrödinger equation, set up the solution so that only the matching of eigenfunctions and their derivatives remain to be done for the following situations:

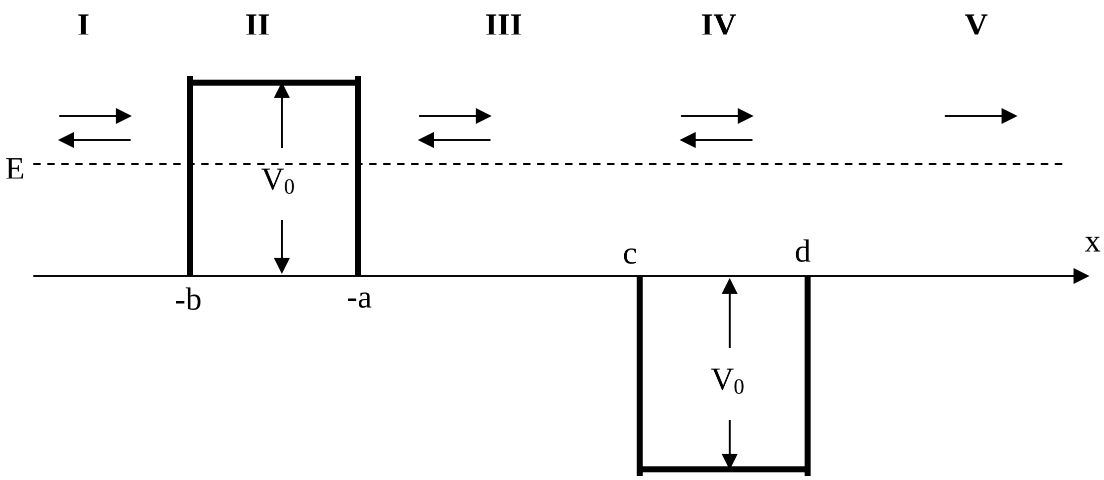


(a) flux $\hbar k/m$ would be incident from the left if the potentials were absent; take $E<V_0$.



(b) with flux of magnitude $\hbar k/m$ incident from the right if the potentials were absent; take $E<V_0$.

Solution:
(a)



Since flux ($f\hbar k/m$) is incident from the left if the potentials were absent, there should not be flux to the left in region V.

For regions I, III, and V: $E = \dfrac{\hbar^2 k^2}{2m} \Rightarrow k = \dfrac{\sqrt{2mE}}{\hbar}$

For region II: $V_0 - E = \dfrac{\hbar^2 \kappa^2}{2m} \Rightarrow \kappa = \dfrac{\sqrt{2m(V_0 - E)}}{\hbar}$

For region IV: $E + V_0 = \dfrac{\hbar^2 \alpha^2}{2m} \Rightarrow \alpha = \dfrac{\sqrt{2m(E + V_0)}}{\hbar}$

With k, κ, and α defined as above:

$\psi_I(x) = Ae^{ikx} + Be^{-ikx}$

$\psi_{II}(x) = Ce^{\kappa x} + De^{-\kappa x}$

$\psi_{III}(x) = Ee^{ikx} + Fe^{-ikx}$

$\psi_{IV}(x) = GAe^{i\alpha x} + He^{-i\alpha x}$

$\psi_V(x) = Le^{ikx}$

Coefficients A, B, C, D, E, F, G, H, and L are to be determined by boundary conditions:

$\psi_I(-b) = \psi_{II}(-b)$

$\psi_I'(-b) = \psi_{II}'(-b)$

$\psi_{II}(-a) = \psi_{III}(-a)$

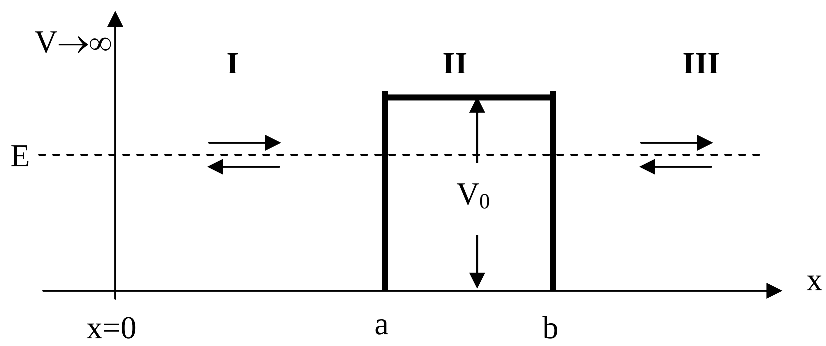
$\psi_{II}'(-a) = \psi_{III}'(-a)$

$\psi_{III}(c) = \psi_{IV}(c)$

$\psi_{III}'(c) = \psi_{IV}'(c)$

$$\psi_{IV}(d) = \psi_V(d)$$
$$\psi_{IV}'(d) = \psi_V'(d)$$

(b)



Since the flux ($\hbar$ k/m) is incident from the right if the potentials were absent, it will be reflected by the wall:

For regions I and III: $E = \dfrac{\hbar^2 k^2}{2m} \Rightarrow k = \dfrac{\sqrt{2mE}}{\hbar}$

For region II: $V_0 - E = \dfrac{\hbar^2 \kappa^2}{2m} \Rightarrow \kappa = \dfrac{\sqrt{2m(V_0 - E)}}{\hbar}$

With k and κ defined as above:

$\psi_I(x) = Ae^{ikx} + Be^{-ikx}$

But because of the wall at x = 0, we require $\psi_I(0) = 0$.

$\therefore A + B = 0 \Rightarrow A = -B$

Therefore we can write

$\psi_I(x) = A \sin kx$

For regions II and III:

$\psi_{II}(x) = Ce^{\kappa x} + De^{-\kappa x}$

$\psi_{III}(x) = Ee^{ikx} + Fe^{-ikx}$

Coefficients A, C, D, E, F are to be determined by boundary conditions:

$$\psi_I(a) = \psi_{II}(a)$$
$$\psi_I'(a) = \psi_{II}'(a)$$
$$\psi_{II}(b) = \psi_{III}(b)$$
$$\psi_{II}'(b) = \psi_{III}'(b)$$