More on unitary transformation

An operator is Unitary if its inverse equal to its adjoints:
\[ U^{-1} = U^+ \quad \text{or} \quad UU^+ = U^+U = I \]

1. Unitary transformation is a linear transformation transforming a vector to another vector in the same space.
2. Unitary transformation transforms an orthonormal basis to another orthonormal basis.
3. Operators have to be transformed also, under similar transformation:
   \[ A' = UAU^{-1} \quad \Rightarrow \quad A' = UAU^+ \]
4. If A is Hermitian, A’ is also Hermitian.
5. A and A’ have the same eigenvalues.
6. Complex numbers remain unchanged under unitary transformation.
   Example: \[ \langle \psi' | \phi' \rangle = \langle \psi | \phi \rangle \]
   \[ \langle \psi' | A' | \phi' \rangle = \langle \psi | A | \phi \rangle \]
8. A unitary transformation does not change the physics of a system; it merely transforms one description of the system to another physically equivalent description.
My convention on Dirac notation

In using Dirac notation, we will adopt the following common understanding:

1. I will use Greek letter to represent a particular. This can be any state vector in the space, not necessary a basis vector or an eigenvector of the operator under consideration. Example: $|\psi\rangle$, $<\phi|$, $|\Psi\rangle$ etc.

2. English or Greek letters with an subscript as index will represent a vector from the basis form by eigenvectors of the operator under consideration. Very often it is the eigenvalue corresponding to the vector (we use eigenvalue as a name of the vector). Example: $|E_n\rangle$ is the n-th vector of the basis form by operator $H$, and $H|E_n\rangle = E_n|E_n\rangle$. $|\alpha_i\rangle$ and $|\alpha_m\rangle$ are the i-th and m-th vectors in the basis.

3. Sometimes when the basis involved is clearly defined, we will just use the subscript index (most commonly used alphabets in this case: l, j, k, l, m, n, o, p, x, y, z) in #2 above as the name and index of the eigenvector in the basis. Very often there is a simple relationship between the alphabet and the eigenvalue, and the alphabet called a quantum number. To avoid confusion, I will try to avoid the use of different alphabets to name two different eigenvectors in the same basis. I will use “prime” and “double prime” etc. to represent different eigenvectors in the same basis. Example: $|m\rangle$ and $|m'\rangle$ are the m-th and m'-th eigenvectors from the same basis, so $<m|m'\rangle = \delta_{mm'}$. $|l\rangle$ and $|m\rangle$ are eigenvectors from two different basis $\{l\}$ and $\{m\}$.

4. In general, look for any “index” information in the notation. If it is indexed, it must be an eigenvector in the basis. If it is not indexed, it is a general vector.
Review of two important properties of eigenstates in Dirac notation

1. The basis is orthonormal:
   \[ < n| n' > = \delta_{nn'} \]

2. The basis is complete:
   \[ \sum_{i=1}^{N} |i><i| = 1 \]