Use of Ladder operators

Let the ladder operators $a$ and $a^+$ for $J_1$, $J_2$, and $J$ be $J_{1-}$, $J_{2-}$, $J_-$ and $J_{1+}$, $J_{2+}$, $J_+$:

$$| j, m_j > = \sum_{m_1', m_2'} | m_1', m_2' > < m_1', m_2' | j m_j >$$

$$\Rightarrow < m_1 m_2 | J_\pm | j, m_j > = \sum_{m_1', m_2'} < m_1 m_2 | J_\pm | m_1' m_2' > < m_1' m_2' | j m_j >$$

$$\Rightarrow < m_1 m_2 | J_\pm | j, m_j > = \sum_{m_1', m_2'} < m_1 m_2 | J_{1\pm} + J_{2\pm} | m_1' m_2' > < m_1' m_2' | j m_j >$$

$$\Rightarrow < m_1 m_2 | j, m_j \pm 1 > \sqrt{j(j+1) - m_j (m_j \pm 1)} \hbar$$

$$= \sum_{m_1', m_2'} [ < m_1 m_2 | m_1', m_2' > < m_1', m_2' | j m_j > \sqrt{j_1 (j_1 + 1) - m_1'(m_1' \pm 1)} \hbar$$

$$+ < m_1 m_2 | m_1', m_2' \pm 1 > < m_1', m_2' | j m_j > \sqrt{j_2 (j_2 + 1) - m_2'(m_2' \pm 1)} \hbar]$$

$$\Rightarrow < m_1 m_2 | j, m_j \pm 1 > \sqrt{j(j+1) - m_j (m_j \pm 1)}$$

$$= [ < m_1 \mp 1, m_2 | j m_j > \sqrt{j_1 (j_1 + 1) - m_1(m_1 \mp 1)} + < m_1, m_2 \mp 1 | j m_j > \sqrt{j_2 (j_2 + 1) - m_2(m_2 \mp 1)}]$$
Strategy in using Ladder operators

1. Start from the top right corner, $m_1 = j_1$, $m_j = j$, and $m_2 = j - j_1$. Let the Clebsch-Gordon coefficient be $C$. Using the step down operator (lower sign) $J_-$, we can figure our all the Clebsch-Gordon coefficient at the right edge of the rectangle.

2. After this, we can use the step up operator (upper sign) $J_+$ to fill up the rectangle column by column, starting from the right edge.

3. At the end, we will get all the Clebsch-Gorden coefficients in terms of $C$. $C$ is to be determined by normalization, or the unitary of the of the transformation matrix.
Spin ½ + Spin ½

Let us consider the case of adding two ½ spins, i.e. \( j_1=1/2 \) and \( j_2=1/2 \). Dimension of the product space is 2x2=4.

We will use the notation to represent these non-interacting basis vectors:

\[
\begin{align*}
|\uparrow\uparrow> & \equiv | m_1=1/2 , m_2=1/2> \\
|\uparrow\downarrow> & \equiv | m_1=1/2 , m_2=-1/2> \\
|\downarrow\uparrow> & \equiv | m_1=-1/2 , m_2=1/2> \\
|\downarrow\downarrow> & \equiv | m_1=-1/2 , m_2=-1/2>
\end{align*}
\]

In interacting representation, \( j=0 \) or 1. If \( j=0 \), \( m_J=0 \). If \( j=1 \), \( m_J=-1, 0, \) or 1. So there are also 4 basis vectors, consistent with the fact that it is a 4 dimensional space. We will use \(|j, m_J>\) to represent these interacting basis vectors:

\[
\begin{align*}
|0, 0> & \equiv | j=0 , m_J=1> \\
|1, 1> & \equiv | j=1 , m_J=1> \\
|1, 0> & \equiv | j=1 , m_J=0> \\
|1, -1> & \equiv | j=1 , m_J=-1>
\end{align*}
\]

These two sets of basis vectors can be expressed as linear combination of each other. To do this, we have to calculate the Clebsch-Gordon coefficients.

|
Clebsch-Gordon coefficients for Spin $\frac{1}{2} +$ Spin $\frac{1}{2}$

Case 1. First consider the case of $j=0$ (so $m_j$ must be 0).

\[
< m_1, m_2 | j, m_j \pm 1 > = \sqrt{j(j+1)-m_j(m_j \pm 1)}
\]

\[
= [< m_1 \mp 1, m_2 | j m_j > \sqrt{j(j+1)-m_j(m_j \mp 1)} + < m_1, m_2 \mp 1 | j m_j > \sqrt{j_2(j_2+1)-m_2(m_2 \mp 1)}]
\]

Upper sign with $m_j = \frac{1}{2}$ \[< \frac{1}{2}, \frac{1}{2} | 0, 1 > \sqrt{0-0} \]

\[= < \frac{1}{2}, \frac{1}{2} | 0, 0 > \sqrt{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} (-\frac{1}{2})} + < \frac{1}{2}, -\frac{1}{2} | 0, 0 > \sqrt{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} (-\frac{1}{2})} \]

\[\Rightarrow < \frac{1}{2}, \frac{1}{2} | 0, 0 > = -< \frac{1}{2}, -\frac{1}{2} | 0, 0 > = -C \]

Hence, $| j, m_j > = | 0, 0 > = C | \frac{1}{2}, \frac{1}{2} > - C | \frac{1}{2}, -\frac{1}{2} >$

\[< 0, 0 | 0, 0 > \Rightarrow C^2 + C^2 = 1 \Rightarrow C = \frac{1}{\sqrt{2}} \]

\[\therefore | 0, 0 > = \frac{1}{\sqrt{2}} | \frac{1}{2}, \frac{1}{2} > - \frac{1}{\sqrt{2}} | \frac{1}{2}, -\frac{1}{2} > = \frac{1}{\sqrt{2}} | \uparrow, \uparrow > - \frac{1}{\sqrt{2}} | \downarrow, \uparrow > \text{ (singlet)} \]
Clebsch-Gordon coefficients for Spin $\frac{1}{2} +$ Spin $\frac{1}{2}$

Case 2. Now the more complicated case of $j=1$ (so $m_J$ can be -1, 0, or 1).

$$|1, 1> = |\uparrow\uparrow>$$

$$|1, 0> = \frac{1}{\sqrt{2}} |\uparrow\downarrow> + \frac{1}{\sqrt{2}} |\downarrow\uparrow>$$

$$|1, -1> = |\downarrow\downarrow>$$
Summary for Spin $\frac{1}{2} + $ Spin $\frac{1}{2}$

\[ j = 0 \text{ (singlet)}: \]
\[
|0,0> = \frac{1}{\sqrt{2}} |\uparrow, \downarrow> - \frac{1}{\sqrt{2}} |\downarrow, \uparrow> \quad \text{Antisymmetric}
\]

\[ j = 1 \text{ (triplet)}: \]
\[
|1,1> = |\uparrow \uparrow>
\]
\[
|1,0> = \frac{1}{\sqrt{2}} |\uparrow \downarrow> - \frac{1}{\sqrt{2}} |\downarrow \uparrow> \quad \text{Symmetric}
\]
\[
|1,-1> = |\downarrow \downarrow>
\]
Pauli exclusion principle

The *total* wavefunction of a system of *Fermions* must be *antisymmetric*.

For two electrons, under LS coupling:

\[ \Psi_{\text{Total}} = \psi_{\text{spatial}} \chi_{\text{spin}} \]

\( \Psi_{\text{Total}} \) has to be antisymmetric (Pauli exclusion principle):

- If \( \psi_{\text{spatial}} \) is symmetric (\( \ell = \text{even} \)) then \( \chi_{\text{spin}} \) is antisymmetric (singlet)
- If \( \psi_{\text{spatial}} \) is antisymmetric (\( \ell = \text{odd} \)) then \( \chi_{\text{spin}} \) is symmetric (triplet)