Problem solving and the scariest exam in the history of undergraduate education

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- Why problem solving?
- Why the old Mathematical Tripos exam?
- Why the falling chain of Hopkins, Tait, Steele and Cayley?
- Why does it conserve energy?
- Conclusions

Why problem solving?

- Keh-fei is a good problem solver
- Scientific research = problem solving
- Can problem solving be taught?

  Optimistically: Yes
- How?

  by the right people (teachers, mentors, peers, students)

  in the right places (institutions, facilities)

  at the right times (post Industrial Revolution)
- Case study:

  The scariest (most competitive) exam in the history of undergraduate education

  = the old Cambridge Mathematical Tripos exam, 1780-1909

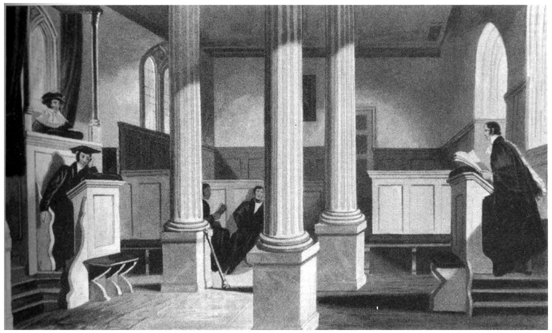
Why the old Cambridge Mathematical Tripos exam?

Model for written competitive exams in the English-speaking world

Cambridge University:

- 800th Anniversary in 2009
- 81 Nobel Prize winners worked or studied there, 1904-2005
- Isaac Newton studied and taught there: 1661-5, 1669-96
- Home of math. Physics in the 19th century: Clerk Maxwell studied and taught there: 1850-4, 1871-9

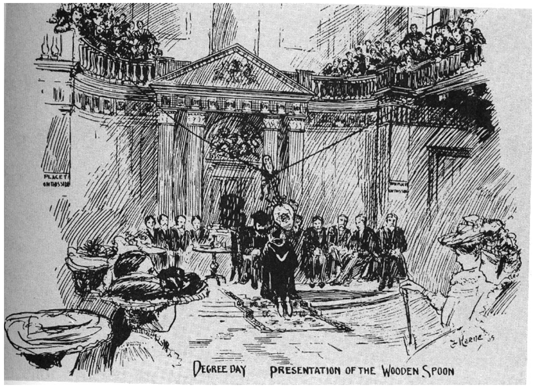
More about the Old Tripos (1):



- Tripos = three-legged stool on which the moderator sat in the original oral disputative exam ("wrangling")
- Old Tripos, 1780-1909:
  Required only for passing with honors
  Written exam
  Graduates ranked in Order of Merit
  Wranglers = students passing with first-class honors:
  Senior Wrangler (SW) = top student
  Second Wrangler (2W) = second best student
  Wooden Spoon (WS) = last student passing with honors
- 1820-1909:
  The old Tripos became increasingly competitive.

More about the Old Tripos (2):

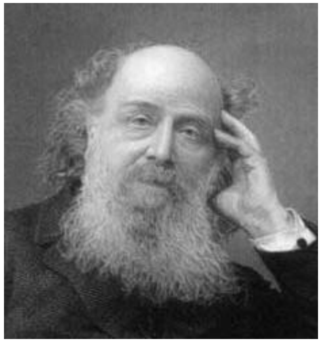
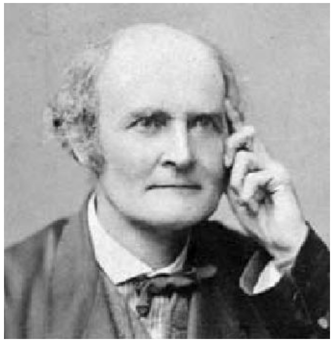
- Lasting 2 weeks:
  1st week: bookwork testing mastery of basic concepts, definitions, laws & proofs using Newton's geometrical method
  2nd week: problem papers often requiring new mathematical techniques applied to novel situations
- Morning and afternoon exam papers,
  each usually 3 hours long:
  Each paper had 10-12 questions. Each question (15-18 minutes each) had several parts: theorem proof, simple application, difficult application. Usually all parts had to be answered correctly to score.
- The 1881 Tripos:
  18 three-hours papers with a total score of 33,541 marks:
  SW got 16,368 marks,
  2W got 13,188,
  WS got 247.
- Examiners believed that marks were proportionate to the abilities of the lower-placed candidates, but did not do full justice to the highest.



Famous wranglers (1):

Only five exceptional mathematicians:

- J.J. Sylvester (2W, 1837)
- A. Cayley (SW, 1842):
  wrote 967 papers, 1841-95
  (Euler wrote only 800 papers)
- W.K. Clifford (2W, 1867):
  Cayley's first student
- G.H. Hardy (4W, 1898):
  collaborated with Ramanujan
- J.E. Littlewood (SW, 1905):
  collaborated with Hardy

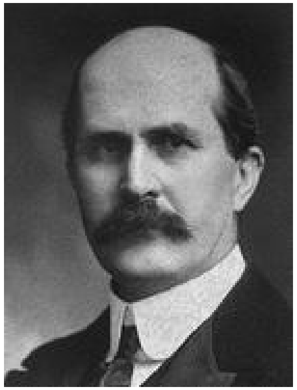




Famous wranglers (2):

Many distinguished physicists thrived in the competitive atmosphere of Cambridge & the fellowship of the Royal Society:



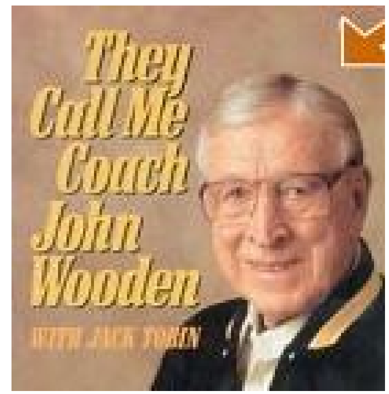


- G. Stokes (SW, 1841):
  fluid dynamics, optics
- W. Thomson (Lord Kelvin) (2W, 1845):
  trans-Atlantic cable, thermodynamics, Kelvin temperature scale
- J. Clerk Maxwell (2W, 1854):
  kinetic theory, Maxwell equations
  (Perhaps greatest physicist after Newton & Einstein)
- J.W. Strutt (Lord Rayleigh) (SW, 1865):
  discovered argon; gases
- J.J. Thomson (2W, 1880):
  discovered electron; electricity in gases
- William Bragg (3W, 1884):
  X-ray crystallography

Coaches of the old Tripos:

Students prepared for the exam by taking private lessons from coaches.



- Coach = tutor/trainer, named after the stagecoach
  (Oxford undergraduate slang, 1830's)
- Coach = closed carriage, named after the Hungarian village Kocs where this carriage originated
- Cambridge Students learned to solve problems against the clock from their coaches.
  (Training in memory and mental discipline.)

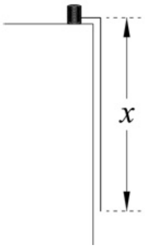
More about the Old Tripos (3):

- Textbooks: were written directly or indirectly for students:
  Whewell (1823), Tait & Steele (1856), Routh (1898).
- Collections of Tripos problems and solutions:
  Wright (1827-31), Wolstenholme (1867, 1878) and private collections of coaches
- Tripos examiners:
  Always top Wranglers, usually young (within 4-8 years of graduation).
  Many were coaches.
- The "arms race", 1820-1909:
  Students were better prepared by coaches and by using textbooks and problem collections, but the problem papers became increasingly more difficult.
  1828 Tripos: 4 days, 8 papers
  1881 Tripos: 9 days, 18 papers
- Tripos stress:
  Cakewalk: Cayley (SW, 1842), Rayleigh (SW, 1865)
  Had enough: Kelvin (2W, 1845)
  Nerves: Maxwell (2W, 1854), J.J. Thomson (2W, 1880)
  Nervous breakdown: James Wilson (SW, 1859)
  Could not walk 50 yards afterwards
  Took 3 months to recuperate & only by forgetting all Cambridge mathematics

The modern math Tripos exams:

- 1909: Order of merit was abandoned because more students avoided the math Tripos by taking the easier Tripos exams in other subjects.
- Present-day Math. Tripos:
  3 years for the BA degree: with four 3-hour papers per year (called Parts IA, IB and II) on different mixed subjects
  Tripos Part III taken in the 4th year: 6 long (9 short) courses, each with a 3-hour (2-hour) exam

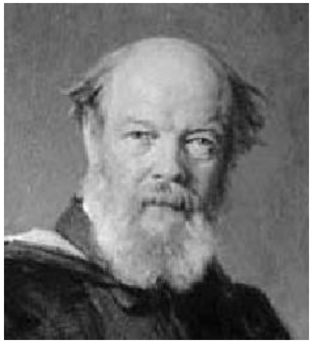
Why the falling chain of Hopkins, Tait, Steele and Cayley?

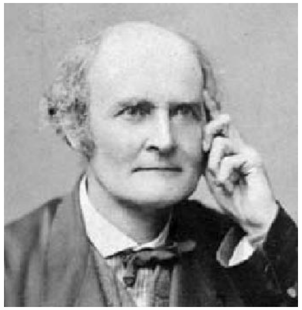


Excessive respect for Newton's methods retarded British mathematics and dynamics in the 18-19th centuries.
It also caused mistakes in Tripos solutions:

Example: Chain falls down link by link from a coil on a table.

Incorrect energy-nonconserving (or inelastic) solution was given by coach William Hopkins, and published by





- Tait and Steele (SW & 2W, 1854), 1856: First textbook on problem
- Cayley (SW, 1842), 1857: First paper on problem
- Wolstenholme (3W, 1850), 1878: Problem collection
- Jeans (2W, 1880), 1907: Textbook
- Lamb (4W, 1898), 1914: Textbook
- Sommerfeld, *Mechanik*, 1943: Textbook
- All the physics and engineering textbooks on mechanics that I have checked

Mistakes in Tripos solutions:

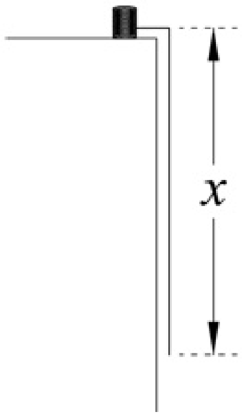
Surprising after scrutiny by generations of teachers, coaches and students

- Hopkins: coached Stokes, Kelvin, Maxwellk, Tait, Todhunter
- Cayley: Perhaps the best pure problem solver in 19th century Cambridge & the best mathematician of Victorian Britain
- Sommerfeld: One of the best teachers of physics

  4 Ph.D. students won Nobel Prizes: Pauli, Heisenberg, Debye, Bethe

  2 postdocs won Nobel Prizes: Pauling, Rabi

Correct energy-conserving (or elastic) solution has been given by



- C.W. Wong and K. Yasui, 2006:

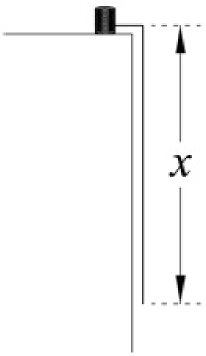
  First correct theoretical treatment

- C.W. Wong, S.H. Youn and K. Yasui, 2007:

  First indirect experimental confirmation

Why does it conserve energy?

Mathematical picture:



Lagrangian:

$$L(x,v) = \frac{\rho}{2} x v^2 + \rho g \frac{x^2}{2}$$

Lagrange equation:

$$\frac{d}{dt}(\rho x v) = \rho x a + \rho v^2$$

$$= \frac{\partial L}{\partial x} = -\rho g x + \frac{s}{2}\rho v^2$$

Solution:

$$a = \frac{g}{3-s} = \begin{cases} g/3 \\ g/2 \end{cases}, \text{ if } \quad s = \begin{cases} 0 & \text{inelastic,} \\ 1 & \text{elastic.} \end{cases}$$

Physical pictures

Inelastic picture of Hopkins, Tait, Steele, Cayley and Sommerfeld:

Falling link sticks to falling chain in a totally inelastic collision (Carnot's theorem)

Elastic picture of Wong and Yasui:

Falling link gains energy when it breaks off from stationary chain segment.

Falling link loses energy when it joins falling chain segment.

The whole process is elastic.

Conclusions:

- Best problem solvers have innate abilities, but are influenced by people, places and times.
  Case study: Famous problem solvers who took the Math Tripos exam
  in Cambridge (home of Newton and Maxwell)

  in Maxwell's time (right after the Industrial Revolution).
- Original research is enhanced by fierce competition
  => The elite world of meritocracy, evolution, capitalism, politics.
  Lesson? To foster competition, only the best students should be graded A+.
- Retention of learning is helped by rewards of pleasure
  => The feel-good world of democracy, socialism, Heaven (the Greatest Society)

  Lesson? To maximize pleasure & memory, every student should be graded A+.
- Grading culture, begun in 19th century Cambridge, flourishes to this day.
- The physics in this talk?
  There is a mistake in the accepted solutions of Tripos problems, after 150 years of careful scrutiny by generations of mathematicians and physicists.