Coherent states for Fermions

We saw that using coherent states for bosons allowed us to make useful and natural semiclassical approximations. For bosons, the normalized coherent states were defined as

$$\boxed{|z\rangle = e^{z b^\dagger} |0\rangle \, e^{-\frac{|z|^2}{2}}} \quad (1)$$

They were eigenstates of b

$$\boxed{b|z\rangle = z|z\rangle} \quad (2)$$

overcomplete

$$\boxed{\langle z_1 | z_2 \rangle = e^{z_1^* z_2 - \frac{1}{2}(|z_1|^2 + |z_2|^2)}} \quad (3)$$

but could be used to form a resolution of the identity

$$\boxed{\mathbb{1} = \int \frac{d^2 z}{\pi} \, |z\rangle\langle z|} \quad (4)$$

We want something similar for fermions. Suppose we start with a single-particle state for which we have a fermion creation operator $\psi^\dagger$ and a destruction operator ψ

$$\boxed{\psi^2 = \psi^{\dagger 2} = 0} \quad (5) \qquad \boxed{\{\psi, \psi^\dagger\} = \psi\psi^\dagger + \psi^\dagger\psi = 1} \quad (6)$$

$\psi^\dagger\psi$ is the number operator, and the empty state is $|0\rangle$, defined by

$$\boxed{\psi|0\rangle = 0} \quad (7) \quad \Rightarrow \quad \boxed{\psi^\dagger\psi|0\rangle = 0} \quad (8)$$

The full state is $\boxed{\psi^\dagger|0\rangle = |1\rangle}$ (9)

$$\boxed{\begin{aligned} \psi^\dagger\psi|1\rangle &= \psi^\dagger\psi\psi^\dagger|0\rangle = \psi^\dagger(\psi\psi^\dagger + \psi^\dagger\psi - \psi^\dagger\psi)|0\rangle \\ &= \psi^\dagger(1-\psi^\dagger\psi)|0\rangle = \psi^\dagger|0\rangle = |1\rangle \end{aligned}} \quad (10)$$

Let us define Grassmann numbers $\eta_1, \eta_2, \cdots$ and $\bar{\eta}_1, \bar{\eta}_2, \cdots$ by

$$\boxed{\eta_i^2 = 0 = \bar{\eta}_i^2} \quad (11) \qquad \boxed{\eta_i\eta_j = -\eta_j\eta_i} \quad (12)$$

$$\boxed{\bar{\eta}_i\bar{\eta}_j = -\bar{\eta}_j\bar{\eta}_i} \quad (13) \qquad \boxed{\eta_i\bar{\eta}_j = -\bar{\eta}_j\eta_i} \quad (14)$$

Further, η's anticommute with the operators $\psi, \psi^\dagger$

$$\boxed{\eta\psi = -\psi\eta} \quad (15) \qquad \boxed{\eta\psi^\dagger = -\psi^\dagger\eta} \quad (16)$$

Now define a fermionic coherent state by

$$\boxed{|\eta\rangle = e^{-\eta\psi^\dagger}|0\rangle = \left[1 - \eta\psi^\dagger + \frac{\eta\psi^\dagger\eta\psi^\dagger}{2!} + \cdots\right]|0\rangle} \quad (17)$$

but $\eta\psi^\dagger\eta\psi^\dagger = -\eta^2\psi^{\dagger 2} = 0$. Similarly all higher powers are zero

$$\boxed{|\eta\rangle = (1-\eta\psi^{+})|0\rangle = |0\rangle - \eta|1\rangle = |0\rangle + |1\rangle\eta} \quad (18)$$

$$\boxed{\langle\bar{\eta}| = \langle 0|(1-\psi\bar{\eta}) = \langle 0| - \langle 1|\bar{\eta} = \langle 0| + \bar{\eta}\langle 1|} \quad (19)$$

$$\boxed{\langle\bar{\eta}|\eta\rangle = 1+\bar{\eta}\eta = e^{\bar{\eta}\eta} = e^{-\eta\bar{\eta}}} \quad (20)$$

Consider

$$\psi|\eta\rangle = \psi|0\rangle - \psi\eta|1\rangle = 0+\eta\psi|1\rangle = \eta|0\rangle.$$

$$= \eta\,(|0\rangle - \eta|1\rangle) \qquad \text{because } \eta^2=0$$

$$\Rightarrow \boxed{\psi|\eta\rangle = \eta|\eta\rangle} \quad (21)$$

Similarly $\boxed{\langle\bar{\eta}|\psi^{+} = \langle\bar{\eta}|\bar{\eta}}$ (22)

In the two-dimensional Hilbert space spanned by $|0\rangle$ and $|1\rangle$ we can consider how to represent the identity. It turns out we need to define integration and differentiation of Grassmann variables. We will define these operations to be as close as possible to the corresponding bosonic ones.

Define $\boxed{\frac{\partial}{\partial\eta}1 = 0}$ (23), $\boxed{\frac{\partial}{\partial\eta}\eta = 1}$ (24) and also that $\frac{\partial}{\partial\eta}$ anticommutes with any other Grassmann

$$\boxed{\left\{\frac{\partial}{\partial\eta}, \eta'\right\} = 0} \quad (25)$$

So, for example

$$\boxed{\frac{\partial}{\partial\eta} e^{-\bar\eta\eta} = \frac{\partial}{\partial\eta}\left[1-\bar\eta\eta\right] = 0 - \frac{\partial}{\partial\eta}\bar\eta\eta = \bar\eta\frac{\partial}{\partial\eta}\eta = \bar\eta} \quad (26)$$

Now, counterintuitively, the most convenient definition of integration turns out to be the same as differentiation

$$\boxed{\int d\eta = 0} \quad (27) \qquad \boxed{\int d\eta\ \eta = 1} \quad (28)$$

and of course $\boxed{\{d\eta, \eta'\} = 0}$ (29)

Now we are ready to resolve the identity

$$\boxed{\mathbb{1} = \int d\bar\eta\, d\eta\ |\eta\rangle\langle\bar\eta|\ e^{-\bar\eta\eta}} \quad (30)$$

Check

$$= \int d\eta\, d\bar\eta\ \left(|0\rangle - \eta|1\rangle\right)\left(\langle 0| - \langle 1|\bar\eta\right)\left(1+\bar\eta\eta\right)$$

$$= \int d\bar\eta\, d\eta\ \left[|0\rangle\langle 0| - |0\rangle\langle 1|\bar\eta + |1\rangle\langle 0|\eta + |1\rangle\langle 1|\eta\bar\eta\right] \otimes (1-\bar\eta\eta)$$

$$= \int d\bar\eta\, d\eta\ \left[|0\rangle\langle 0|(1-\bar\eta\eta) - |0\rangle\langle 1|\bar\eta - |1\rangle\langle 0|\eta + |1\rangle\langle 1|\eta\bar\eta\right]$$

Now $\int d\bar\eta\, d\eta = 0 \qquad \int d\bar\eta\, d\eta\, \eta = \int d\bar\eta\, d\eta\, \bar\eta = 0$

$$\int d\bar\eta\, d\eta\, (-)\bar\eta\eta = -\int d\bar\eta\, d\eta\, \bar\eta\eta = \int d\bar\eta\, d\eta\, \eta\bar\eta = 1$$

So $$\boxed{\int d\bar{\eta}\, d\eta\, |\eta\rangle\langle\bar{\eta}|\, e^{-\bar{\eta}\eta} = |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}} \quad (31)$$

Now consider an arbitrary operator

$$\boxed{\hat{A} = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} = A_{00} + (A_{11} - A_{00})\psi^{+}\psi + A_{01}\psi + A_{10}\psi^{+}} \quad (32)$$

Let us see how this acts on $|\eta\rangle$

$$\boxed{\begin{aligned} \hat{A}|\eta\rangle = \hat{A}(|0\rangle + |1\rangle\eta) &= A_{00}|0\rangle + A_{10}|1\rangle + (A_{01}|0\rangle + A_{11}|1\rangle)\eta \\ &= (A_{00} + A_{01}\eta)|0\rangle + (A_{10} - A_{11}\eta)|1\rangle \end{aligned}} \quad (33)$$

However, there is another way of expressing this. Start with

$$\boxed{\hat{A}|\eta\rangle = \mathbb{1}\hat{A}|\eta\rangle = \int d\bar{\eta}'\, d\eta'\, e^{-\bar{\eta}'\eta'}\, |\eta'\rangle\langle\bar{\eta}'|\hat{A}|\eta\rangle} \quad (34)$$

So we need $\langle\bar{\eta}'|\hat{A}|\eta\rangle$

$$\boxed{\begin{aligned} \langle\bar{\eta}'|\hat{A}|\eta\rangle &= (\langle 0| - \langle 1|\bar{\eta}')\,\hat{A}\,(|0\rangle - \eta|1\rangle) \\ &= (\langle 0| + \bar{\eta}'\langle 1|)\,\hat{A}\,(|0\rangle + |1\rangle\eta) \end{aligned}} \quad (35)$$

$$\boxed{\mathcal{A}(\bar{\eta}', \eta) = A_{00} + A_{01}\eta + \bar{\eta}' A_{10} + \bar{\eta}'\eta\, A_{11}} \quad (36)$$

This is the integral kernel of the operator $\hat{A}$ in the coherent state representation.

Let us check that this is consistent with (33)

$$\hat{A}|\eta\rangle = \int d\bar{\eta}' d\eta' \, (1-\bar{\eta}'\eta') \left[|0\rangle - \eta' |1\rangle\right]$$
$$\otimes \left[A_{00} + A_{01}\eta + \bar{\eta}' A_{10} + \bar{\eta}'\eta \, A_{11}\right]$$

Recall $\eta' |1\rangle = -|1\rangle \eta'$

$$\hat{A}|\eta\rangle = |0\rangle \int d\bar{\eta}' d\eta' \, (1-\bar{\eta}'\eta') \left[A_{00} + A_{01}\eta + \bar{\eta}' A_{10} + \bar{\eta}'\eta A_{11}\right]$$
$$+ |1\rangle \int d\bar{\eta}' d\eta' (1-\bar{\eta}'\eta') \, \eta' \left[A_{00} + A_{01}\eta + \bar{\eta}' A_{10} + \bar{\eta}'\eta A_{11}\right]$$

Use the rules of integration

$$\hat{A}|\eta\rangle = |0\rangle \left(A_{00} + A_{01}\eta\right) + |1\rangle \left(A_{10} + \eta A_{11}\right)$$
$$= (A_{00} + A_{01}\eta) |0\rangle + (A_{10} - \eta A_{11}) |1\rangle$$

Same as (33)

We need to know the trace of a matrix in the coherent state representation.

Let us write

$$\boxed{\hat{A} = \mathbb{1} \hat{A} \mathbb{1} = \int d\bar{\eta}_1 d\eta_1 \, d\bar{\eta}_2 d\eta_2 \, e^{-\bar{\eta}_1\eta_1} e^{-\bar{\eta}_2\eta_2} \, |\eta_2\rangle \langle\bar{\eta}_2| \hat{A} |\eta_1\rangle \langle\bar{\eta}_1| \qquad (37)}$$

$$\boxed{\hat{A} = \int d\bar{\eta}_1 d\eta_1 d\bar{\eta}_2 d\eta_2 \, (1-\bar{\eta}_1\eta_1)(1-\bar{\eta}_2\eta_2) \otimes |\eta_2\rangle \left[A_{00} + A_{01}\eta_1 + A_{10}\bar{\eta}_2 + A_{11}\bar{\eta}_2\eta_1 \right] \langle\bar{\eta}_1|} \quad (38)$$

We can do the $\eta_1, \bar{\eta}_2$ integrations

$d\eta_1 d\bar{\eta}_2$ is bosonic, that is, it commutes with all Grassmann variables, including $|\eta_2\rangle$. Similarly $(1-\bar{\eta}_1\eta_1)$ is bosonic

$$\hat{A} = \int d\bar{\eta}_1 d\eta_2 \, |\eta_2\rangle \int d\eta_1 d\bar{\eta}_2 \left[1 - \bar{\eta}_1\eta_1 - \bar{\eta}_2\eta_2 + \bar{\eta}_1\eta_1\bar{\eta}_2\eta_2\right] \left[A_{00} + A_{01}\eta_1 + A_{10}\bar{\eta}_2 + A_{11}\bar{\eta}_2\eta_1\right] \langle\bar{\eta}_1|$$

In order to survive integration the term must be $\bar{\eta}_2\eta_1$ and nothing else. So

$$= \int d\bar{\eta}_1 d\eta_2 \, |\eta_2\rangle \int d\eta_1 d\bar{\eta}_2 \left[A_{00}\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 - A_{01}\bar{\eta}_2\eta_2\eta_1 - A_{10}\bar{\eta}_1\eta_1\bar{\eta}_2 + A_{11}\bar{\eta}_2\eta_1 \right] \langle\bar{\eta}_1|$$

$$\boxed{\hat{A} = \int d\bar{\eta}_1 d\eta_2 \, |\eta_2\rangle \left[-A_{00}\bar{\eta}_1\eta_2 + A_{01}\eta_2 + A_{10}\bar{\eta}_1 + A_{11} \right] \langle\bar{\eta}_1|} \quad (39)$$

Now taking the trace is usually done by "setting the last index = the 1st index"

So let us try closing the bracket by

$$|\eta_2\rangle\langle\bar{\eta}_1| \Rightarrow \langle\bar{\eta}_1|\eta_2\rangle = e^{\bar{\eta}_1\eta_2}$$

$$\Rightarrow \boxed{\begin{aligned}&\int d\bar\eta_1\, d\eta_2 \; (1+\bar\eta_1\eta_2)\left[A_{00}\eta_2\bar\eta_1 + A_{01}\eta_2 + A_{10}\bar\eta_1 + A_{11}\right] \quad (40)\\ &= A_{00} - A_{11} \qquad \text{NOT THE TRACE!}\end{aligned}}$$

So we are off by a sign if we do the simplest way of closing the bracket. The way to fix this is to let $\boxed{\eta_2 \to -\eta_2}$ (41) when we close the bracket

$$|\eta_2\rangle\langle\bar\eta_1| \Rightarrow \boxed{\langle\bar\eta_1|-\eta_2\rangle = e^{-\bar\eta_1\eta_2}} \quad (42)$$

This will be important in determining the boundary conditions for fermions in imaginary time to be antiperiodic

$$\boxed{\psi(\beta) = -\psi(0)} \quad (43)$$

Now let us introduce many fermionic degrees of freedom with a quadratic action.

The most common situation is a hermitian "Hamiltonian" with both ψ_i and ψ_i^+

$$\boxed{H = \psi_i^+ h_{ij}\psi_j} \quad (44) \qquad \boxed{h_{ij} = h_{ji}^*} \quad (45)$$

Consider now a toy "partition function"

$$\boxed{Z = \int \prod_i \left(d\bar\eta_i\, d\eta_i\right) e^{-\bar\eta_i h_{ij}\eta_j}} \quad (46)$$

Let us make a unitary transformation to diagonalize h

$$\boxed{h = U^{\dagger} h_D U \qquad \text{where} \quad h_{D\,ij} = \varepsilon_i \delta_{ij}} \quad (47)$$

define $\quad \boxed{\chi_i = U_{ij} \eta_j} \quad (48) \qquad \boxed{\bar{\chi}_j = \bar{\eta}_i U^{\dagger}_{ij}} \quad (49)$

What is the "Jacobian" for Grassmann variables? Recall that integration is the same as differentiation, so funny stuff happens

If $\quad \boxed{\chi = \alpha \eta} \quad (50) \qquad \alpha$ commutes (is bosonic)

We want to preserve

$$\boxed{\int d\eta\, \eta = 1 = \int d\chi\, \chi = \int d\chi\, \alpha \eta} \quad (51)$$

$$\Rightarrow \qquad \boxed{d\eta = \alpha\, d\chi} \quad (52)$$

Note that this is the opposite of the Jacobian rule for normal (bosonic) integrals

For bosonic variables x, y if $\boxed{y = \alpha x}$ (53) then $\boxed{dx = \alpha^{-1} dy}$ (54). This is just one of the ways in which Grassmann variables are wierd

Now let us go to many variables.

$$\boxed{\chi_i = U_{ij} \eta_j} \quad (55) \qquad j = 1, \cdots n$$

$$\boxed{\begin{aligned}\int d\eta_1 d\eta_2 \cdots d\eta_n \, \eta_n \eta_{n-1} \cdots \eta_1 &= 1 \\ &= \int d\chi_1 d\chi_2 \cdots d\chi_n \, \chi_n \chi_{n-1} \cdots \chi_1 \\ &= \int d\chi_1 d\chi_2 \cdots d\chi_n \left[U_{n j_n} \eta_{j_n}\right]\left[U_{n-1, j_{n-1}} \eta_{j_{n-1}}\right] \cdots \left[U_{1 j_1} \eta_{j_1}\right]\end{aligned}} \quad (56)$$

Since the η's all anticommute and so do the χ's clearly the numerical factor is

$$\boxed{\begin{aligned}&\frac{1}{n!} \epsilon_{i_1 i_2 \cdots i_n} U_{i_1 j_1} U_{i_2 j_2} \cdots U_{i_n j_n} \epsilon_{j_1 j_2 \cdots j_n} \\ &= \det U\end{aligned}} \quad (57)$$

So the Jacobian rule is

$$\boxed{\prod_1^n d\eta_j = \prod_1^n d\chi_j \det U} \quad (58)$$

again the opposite of the rule for normal bosonic integrals.

For our case, the matrix U is Unitary

$$\boxed{|\det U| = 1} \quad (59)$$

Also, we have not only η_i but also $\bar{\eta}_i$. Recall

$$\chi_i = U_{ij} \eta_j \qquad \bar{\chi}_j = \bar{\eta}_i U^\dagger_{ij}$$

$$\Rightarrow \boxed{\prod_{i=1}^{n} (d\bar{\eta}_i \, d\eta_i) = \prod_{i=1}^{n} (d\bar{\chi}_i \, d\chi_i) \quad \det U \, \det U^{+} \\ = \prod_{i=1}^{n} (d\bar{\chi}_i \, d\chi_i)} \quad (60)$$

$$\text{So} \quad Z = \int \prod_{i=1}^{n} (d\bar{\eta}_i \, d\eta_i) \, e^{-\bar{\eta}_i h_{ij} \eta_j}$$

$$= \int \prod_{i=1}^{n} (d\bar{\chi}_i \, d\chi_i) \, e^{-\varepsilon_i \bar{\chi}_i \chi_i}$$

$$\boxed{Z = \prod_{i=1}^{n} \int d\bar{\chi}_i \, d\chi_i \, (1 - \varepsilon_i \bar{\chi}_i \chi_i) = \prod_{i=1}^{n} \varepsilon_i = \det(h)} \quad (61)$$

A similar analysis for bosonic complex scalars ϕ_i $\bar{\phi}_i$ gives the opposite result

$$\boxed{Z_{boson} = \int \prod d\bar{\phi}_i \, d\phi_i \, e^{-\bar{\phi}_i h_{ij} \phi_j} = \frac{1}{\det(h)}} \quad (62)$$

If we have bosons and fermions with the same h, at the quadratic level

$$\boxed{Z = Z_f \, Z_b = 1 = e^{-\beta F} \quad \Rightarrow \quad F = 0} \quad (63)$$

This is the most elementary form of supersymmetry. It is attractive because the free energy, or vacuum energy in QFT is zero, which leads to an approximate solution of the cosmological constant problem

Consider now correlators in this toy model such as

$$\boxed{\langle \eta_i \, \bar\eta_j \rangle = \frac{1}{Z} \left\{ \int \prod_1^n d\bar\eta_\ell \, d\eta_\ell \right\} e^{-\bar\eta_\ell h_{\ell m} \eta_m} \eta_i \bar\eta_j} \quad (64)$$

Go to the representation where h is diagonal

$$\boxed{\eta_i = U^\dagger_{ir} \chi_r} \quad (65) \qquad \boxed{\bar\eta_j = \bar\chi_s U_{sj}} \quad (66)$$

$$\boxed{\langle \eta_i \bar\eta_j \rangle = \frac{1}{Z} \left\{ \int \prod_1^n d\bar\chi_\ell \, d\chi_\ell \right\} e^{-\varepsilon_m \bar\chi_m \chi_m} U^\dagger_{ir} U_{sj} \, \chi_r \bar\chi_s} \quad (67)$$

Clearly, unless $r = s$ the average vanishes. For $r = s$, all variables other than r give the same integral, but the r integral is now different

$$\boxed{\int d\bar\chi_r \, d\chi_r \quad \chi_r \bar\chi_r \, (1 - \varepsilon_r \bar\chi_r \chi_r) = 1} \quad (68)$$

No sum on r

as compared to ε_r when one considers Z.

$$\boxed{\int d\bar\chi_r \, d\chi_r \, (1 - \varepsilon_r \bar\chi_r \chi_r) = \varepsilon_r} \quad (69)$$

$$\Rightarrow \boxed{\begin{aligned} \langle \eta_i \bar\eta_j \rangle &= U^\dagger_{ir} \frac{1}{\varepsilon_r} U_{rj} = \sum_r \langle i | r \rangle \frac{1}{\varepsilon_r} \langle r | j \rangle \\ &= h^{-1}_{ij} \end{aligned}} \quad (70)$$

Now we are ready to do a finite-T quantum problem. Consider the fermionic Hubbard model on a d-dimensional lattice

$$\boxed{\hat{K} = \hat{H} - \mu\hat{N} = -t \sum_{\bar{x},\hat{e}_i,s} \left\{ c_s^+(\bar{x}+\hat{e}_i)\, c_s(\bar{x}) + c_s^+(\bar{x})\, c_s(\bar{x}+\hat{e}_i) \right\} + \sum_{\bar{x}}' \left\{ -\mu \sum_s c_s^+(\bar{x})\, c_s(\bar{x}) + \frac{U}{2}\left[\sum_s c^+(\bar{x})\, c(\bar{x})\right]^2 \right\}} \quad (71)$$

$s = \uparrow, \downarrow$ stands for spin. The c, c^+ are canonical fermion operators

$$\boxed{\begin{aligned} \{c_s(\bar{x}), c_{s'}(\bar{x}')\} &= 0 = \{c_s^+(\bar{x}), c_{s'}^+(\bar{x}')\} \\ \{c_s(\bar{x}), c_{s'}^+(\bar{x}')\} &= \delta_{ss'}\, \delta_{\bar{x},\bar{x}'} \end{aligned}} \quad (72)$$

The partition f^n of this model is

$$\boxed{Z = \mathrm{Tr}\left\{ e^{-\beta\hat{K}} \right\}} \quad (73)$$

As usual do the time-slicing

$$\boxed{Z = \mathrm{Tr}\left\{ \prod_{j=1}^{N} e^{-\delta\tau\,\hat{K}_j} \right\}} \quad (74) \qquad \boxed{\delta\tau = \beta/N} \quad (75)$$

Insert a complete set of states between each two adjacent slices. There are many ways of resolving the identity, and we will use (30).

Let us simplify to a problem with only one site but keep both spin orientations.

$$\boxed{\hat{K} = (\varepsilon-\mu)c_s^\dagger c_s + \frac{U}{2}(c_s^\dagger c_s)^2} \quad (76)$$

Implicit sum over s

1st normal-order the interaction

$$(c_s^\dagger c_s)^2 = c_s^\dagger c_s \; c_{s'}^\dagger c_{s'}$$

Implicit sums over s, s'

$$= c_s^\dagger \left(\{c_s, c_{s'}^\dagger\} - c_{s'}^\dagger c_s \right) c_{s'}$$

$$= \delta_{ss'} c_s^\dagger c_{s'} - c_s^\dagger c_{s'}^\dagger c_s c_{s'}$$

or $$\boxed{(c_s^\dagger c_s)^2 = c_s^\dagger c_s + c_s^\dagger c_{s'}^\dagger c_{s'} c_s} \quad (77)$$

Now we introduce the Grassmann variables $\psi_s(i)$ $\bar{\psi}_s(i)$ for the i^{th} time slice

$$Z = \int \prod_{i=1}^{N} \left(d\bar{\psi}_\uparrow(i)\, d\psi_\uparrow(i)\, d\bar{\psi}_\downarrow(i)\, d\psi_\downarrow(i) \right) e^{-\sum_{i=1}^{N} \bar{\psi}_s(i)\psi_s(i)}$$

$$\langle \bar{\psi}_s(N) | (1-\delta\tau \hat{K}) | \psi_s(N-1)\rangle \langle \bar{\psi}_s(N-1) | (1-\delta\tau\hat{K}) | \psi_s(N-2)\rangle \cdots$$

$$\cdots \langle \bar{\psi}_s(2) | (1-\delta\tau\hat{K}) | \psi_s(1)\rangle \underbrace{\langle \bar{\psi}_s(1) | -\psi_s(N)\rangle} \quad (78)$$

VERY IMPORTANT

The last factor is the "closing" of the trace and has the minus sign we saw was necessary in (40), (41)

$$\boxed{\langle \bar{\psi}_s(1) | -\psi_s(N)\rangle = e^{-\bar{\psi}_s(1)\psi_s(N)}} \quad (79)$$

So, we need to find

$$\langle \bar{\psi}_s(i+1) | (1-\delta\tau \hat{K}) | \psi_s(i) \rangle$$

Recall that $|\psi_s\rangle$ was designed to be an eigenstate of c_s with eigenvalue ψ_s

$$c_s |\psi_s\rangle = \psi_s |\psi_s\rangle$$

$$\langle \bar{\psi}_s | c_s^+ = \langle \bar{\psi}_s | \bar{\psi}_s$$

$$\Rightarrow \langle \bar{\psi}_s(i+1) | \left\{ 1-\delta\tau \left[(\varepsilon-\mu) c_s^+ c_s + \frac{U}{2} c_s^+ c_{s'}^+ c_{s'} c_s \right] \right\} |\psi_s(i)\rangle$$

$$= \langle \bar{\psi}_s(i+1) | \psi_s(i) \rangle \left\{ 1-\delta\tau \left[(\varepsilon-\mu) \bar{\psi}_s(i+1) \psi_s(i) + \frac{U}{2} \bar{\psi}_s(i+1) \bar{\psi}_{s'}(i+1) \psi_{s'}(i) \psi_s(i) \right] \right\}$$

From (20)

$$\langle \bar{\psi}_s(i+1) | \psi_s(i) \rangle = e^{\bar{\psi}_s(i+1) \psi_s(i)}$$

So

$$\boxed{\langle \bar{\psi}_s(i+1) | [1-\delta\tau \hat{K}] | \psi_s(i) \rangle = e^{\bar{\psi}_s(i+1)\psi_s(i)} \, e^{-\delta\tau \left[(\epsilon-\mu) \bar{\psi}_s(i+1)\psi_s(i) + \frac{U}{2} \bar{\psi}_s(i+1) \bar{\psi}_{s'}(i+1) \psi_{s'}(i) \psi_s(i) \right]}} \quad (80)$$

Now put all this together

$$\boxed{Z = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \; e^{\sum_{i=1}^{N} (\bar{\psi}_s(i+1) - \bar{\psi}_s(i)) \psi_s(i)} \otimes e^{-\delta\tau \sum_i \left[(\epsilon-\mu) \bar{\psi}_s(i+1)\psi_s(i) + \frac{U}{2} \bar{\psi}_s(i+1) \bar{\psi}_{s'}(i+1) \psi_{s'}(i) \psi_s(i) \right]}} \quad (81)$$

where $$\bar{\psi}_s(N+1) = -\bar{\psi}_s(1) \,, \quad \psi_s(N+1) = -\psi_s(1) \tag{82}$$

Now take the limit $\delta\tau \to 0$ and rewrite

$$\sum_{i=1}^{N} \left(\bar{\psi}_s(i+1) - \bar{\psi}_s(i)\right) \psi_s(i) = \sum_{i=1}^{N} \delta\tau \, \frac{\partial \bar{\psi}_s}{\partial \tau} \, \psi_s \Rightarrow \int_0^\beta d\tau \, \frac{\partial \bar{\psi}_s}{\partial \tau} \psi_s \tag{83}$$

$$Z = \int \mathcal{D}\bar{\psi}_s \mathcal{D}\psi_s \, e^{-S} \tag{84}$$

$$S = \int_0^\beta d\tau \left\{ -\frac{\partial \bar{\psi}_s(\tau)}{\partial \tau} \psi_s(\tau) + (\varepsilon - \mu) \bar{\psi}_s(\tau) \psi_s(\tau) + \frac{U}{2} \bar{\psi}_s(\tau) \bar{\psi}_{s'}(\tau) \psi_{s'}(\tau) \psi_s(\tau) \right\} \tag{85}$$

with the important antiperiodic boundary conditions

$$\psi_s(\beta) = -\psi_s(0) \qquad \bar{\psi}_s(\beta) = -\bar{\psi}_s(0) \tag{86}$$

Now consider correlation functions. The most natural correlators in this imaginary time path-integral language are Time-ordered correlators

This is the standard convention

$$\mathcal{G}_{ss'}(\tau_1,\tau_2) = \langle T_\tau \left(c_s(\tau_1) \, c^{+}_{s'}(\tau_2) \right) \rangle = -\Theta(\tau_1 - \tau_2) \langle c_s(\tau_1) \, c^{+}_{s'}(\tau_2) \rangle + \Theta(\tau_2 - \tau_1) \langle c^{+}_{s'}(\tau_2) \, c_s(\tau_1) \rangle \tag{87}$$

where $\boxed{c_s(\tau) = e^{\hat{K}\tau} c_s e^{-\hat{K}\tau}}$ (88) and for $\tau_1 > \tau_2$

$$\boxed{\langle c_s(\tau_1)\, c^{+}_{s'}(\tau_2)\rangle = \frac{1}{Z} \mathrm{Tr}\left\{ e^{\hat{K}\tau_1} c_s e^{-\hat{K}\tau_1} e^{\hat{K}\tau_2} c^{+}_{s'} e^{-\hat{K}\tau_2} e^{-\beta\hat{K}} \right\}} \quad (89)$$

The reason is that our time-slicing is putting earlier times to the right and later times to the left, and the operators that appear in the correlator have to be in their own time-slice.

This is the same reason that the Feynman propagator appears naturally in the real-time path integral.

Now, we also know that we must have time-translation symmetry in τ. In fact, for $\tau_1 > \tau_2$

$$\boxed{\langle c_s(\tau_1)\, c^{+}_{s'}(\tau_2)\rangle = \frac{1}{Z} \mathrm{Tr}\left\{ e^{\hat{K}\tau_1} c_s e^{-\hat{K}(\tau_1-\tau_2)} c^{+}_{s'} e^{-\hat{K}\beta} e^{-\hat{K}\tau_2} \right\}} \quad (90)$$

By the cyclic invariance of the trace move the last factor to the front

$$\boxed{\begin{aligned}\langle c_s(\tau_1)\, c^{+}_{s'}(\tau_2)\rangle &= \frac{1}{Z} \mathrm{Tr}\left\{ e^{\hat{K}(\tau_1-\tau_2)} c_s e^{-\hat{K}(\tau_1-\tau_2)} c^{+}_{s'} e^{-\hat{K}\beta} \right\} \\ &= \langle c_s(\tau_1-\tau_2)\, c^{+}_{s'}(0)\rangle\end{aligned}} \quad (91)$$

which shows the time-translation property. Furthermore, it turns out that $G_{ss'}(\tau)$ is also antiperiodic when $\tau \to \tau+\beta$

$$\boxed{\mathcal{G}_{ss'}(\tau+\beta) = -\mathcal{G}_{ss'}(\tau)} \quad (92)$$

which follows from the same property for $\psi_s, \bar{\psi}_s$ Eq (86). In terms of Grassmann integrals

$$\boxed{\begin{aligned}\mathcal{G}_{ss'}(\tau_1,\tau_2) &= \frac{1}{Z}\int \mathcal{D}\bar{\psi}_s \mathcal{D}\psi_s \left(T_\tau \psi_s(\tau_1)\bar{\psi}_{s'}(\tau_2)\right) e^{-S} \\ &= \mathcal{G}_{ss'}(\tau_1-\tau_2)\end{aligned}} \quad (93)$$

To show 1st set $\tau_2=0$ and $0<\tau_1<\beta$ then

$$\boxed{T_\tau\left(\psi_s(\tau_1)\bar{\psi}_{s'}(0)\right) = -\psi_s(\tau_1)\bar{\psi}_{s'}(0)} \quad (94)$$

If $\tau_2=\beta$ then $\boxed{T_\tau\left(\psi_s(\tau_1)\bar{\psi}_{s'}(\beta)\right) = +\bar{\psi}_{s'}(\beta)\psi_s(\tau_1)}$ (95)

Using (86) $\bar{\psi}_{s'}(\beta) = -\psi_{s'}(0)$, so

$$\boxed{T_\tau\left(\psi_s(\tau_1)\bar{\psi}_{s'}(\beta)\right) = -\bar{\psi}_{s'}(0)\psi_s(\tau_1) = \psi_s(\tau_1)\bar{\psi}_{s'}(0) = -T_\tau\left(\psi_s(\tau_1)\bar{\psi}_{s'}(0)\right)} \quad (96)$$

Now we generalize to the d-dimensional Hubbard model

$$\boxed{\begin{aligned}S = \int_0^\beta d\tau \Big\{ \sum_{\bar{x}} \Big[&\bar{\psi}_s(\bar{x},\tau)\frac{\partial}{\partial\tau}\psi_s(\bar{x},\tau) - \mu\bar{\psi}_s(\bar{x},\tau)\psi_s(\bar{x},\tau) \\ &+ \frac{U}{2}\bar{\psi}_s(\bar{x},\tau)\bar{\psi}_{s'}(\bar{x},\tau)\psi_{s'}(\bar{x},\tau)\psi_s(\bar{x},\tau)\Big] \\ &- t\sum_{\vec{x},\hat{e}_i}\left(\bar{\psi}_s(\bar{x}+\hat{e}_i,\tau)\psi_s(\bar{x},\tau) + \bar{\psi}_s(\bar{x},\tau)\psi_s(\bar{x}+\hat{e}_i,\tau)\right)\Big\}\end{aligned}} \quad (97)$$

Let us start with the simplest case, the non-interacting model ($U=0$). Then, a fourier transform

$$\boxed{\psi_s(\bar{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\bar{k}\cdot\bar{x}} \psi_s(\bar{k})} \quad (98)$$

$$\boxed{\bar{\psi}_s(\bar{x}) = \int \frac{d^d k}{(2\pi)^d} e^{-i\bar{k}\cdot\bar{x}} \bar{\psi}_s(\bar{k})} \quad (99)$$

diagonalizes $\hat{\mathcal{H}}_0$

$$\boxed{S_0 = \int_0^\beta d\tau \int \frac{d^d k}{(2\pi)^d} \left\{ \bar{\psi}_s(\bar{k},\tau) \frac{\partial}{\partial \tau} \psi_s(\bar{k},\tau) + (\varepsilon(\bar{k})-\mu) \bar{\psi}_s(\bar{k},\tau) \psi_s(\bar{k},\tau) \right\}} \quad (100)$$

where $\boxed{\varepsilon(\bar{k}) = -2t\left(\cos k_1 + \cos k_2 + \cdots + \cos k_d\right)}$ (101)

Let us now fourier transform in τ as well

$$\boxed{\begin{aligned} \psi_s(\bar{k},\tau) &= \frac{1}{\beta} \sum_{ip_n} e^{-ip_n\tau} \psi_s(\bar{k}, ip_n) \\ \bar{\psi}_s(\bar{k},\tau) &= \frac{1}{\beta} \sum_{ip_n} e^{ip_n\tau} \bar{\psi}_s(\bar{k}, ip_n) \end{aligned}} \quad (102)$$

The antiperiodic b.c. on ψ requires

$$\boxed{e^{ip_n\beta} = -1 \Rightarrow p_n = \frac{(2n+1)\pi}{\beta}} \quad (103)$$

Now $$\int_0^\beta d\tau \quad \bar{\psi}_s(\bar{k},\tau)\frac{\partial}{\partial\tau}\psi_s(\bar{k},\tau)$$ (104)

$$= \frac{1}{\beta^2}\int_0^\beta d\tau \sum_{ip_n\, ip_n'} e^{ip_n'\tau - ip_n\tau}\,\bar{\psi}_s(\bar{k},ip_n')(-ip_n)\,\psi_s(\bar{k},ip_n)$$

$$\int_0^\beta d\tau\, e^{i\tau(p_n'-p_n)} = \int_0^\beta d\tau\, e^{i\tau 2\pi(n'-n)/\beta} = \beta\,\delta_{n,n'}$$ (105)

So the action becomes (106)

$$S_0 = \frac{1}{\beta}\sum_{ip_n}\int\frac{d^dk}{(2\pi)^d}\,\bar{\psi}_s(\bar{k},ip_n)\,\psi_s(\bar{k},ip_n)\left[-ip_n+\varepsilon(\bar{k})-\mu\right]$$

Now this is diagonal in p_n and $\bar{k}$ so we can do the integral. Using (61) we get

$$Z = \prod_{p_n,\vec{k}}\left[-ip_n+\varepsilon(\bar{k})-\mu\right] = e^{-\beta F}$$ (107)

$$\Rightarrow F = -\frac{1}{\beta}\sum_{ip_n,\bar{k}}\ln\left[-ip_n+\varepsilon(\bar{k})-\mu\right]$$

$$F = -\frac{1}{\beta}L^d\sum_{ip_n}\int\frac{d^dk}{(2\pi)^d}\ln\left[-ip_n+\varepsilon(\bar{k})-\mu\right]$$ (108)

Let us calculate the occupation of a state at $\bar{k}$ as our simplest example of a nontrivial correlation f^n.

We want $\boxed{\langle c^{+}(\tau+\epsilon,\bar{k})\; c(\tau,\bar{k})\rangle \quad \epsilon\to 0^{+}}$ (109)

ϵ is a positive infinitesimal because in order to make sure that we are computing $c^{+}c$ and not cc^{+} we need to have c^{+} at a slightly later time than c.

$$\boxed{\langle c^{+}(\tau+\eta,\bar{k})\, c(\tau,\bar{k})\rangle = \frac{1}{\beta^{2}}\sum_{\substack{ip_{n_1}\\ ip_{n_2}}} \langle \bar{\psi}(ip_{n_1},\bar{k})\,\psi(ip_{n_2},\bar{k})\rangle\, e^{ip_{n_1}(\tau+\epsilon)}\, e^{-ip_{n_2}\tau}} \quad (110)$$

From (70) and the diagonal action (106) we know

$$\boxed{\langle \bar{\psi}(ip_{n_1},\bar{k})\;\psi(ip_{n_2},\bar{k})\rangle = -\frac{\beta\,\delta_{p_{n_1},p_{n_2}}}{-ip_{n_1}+\varepsilon(\bar{k})-\mu}} \quad (111)$$

So, we need to do

$$\boxed{\frac{1}{\beta}\sum_{ip_n}\frac{e^{ip_n\epsilon}}{ip_n-(\varepsilon(\bar{k})-\mu)}} \quad (112) \qquad \epsilon\to 0^{+}$$

Recall $\quad ip_n = \frac{(2n+1)\pi}{\beta}$

Here is how to do such sums. Consider the following function of a complex variable z

$$\boxed{n_F(z) = \frac{1}{e^{\beta z}+1}} \quad (113)$$

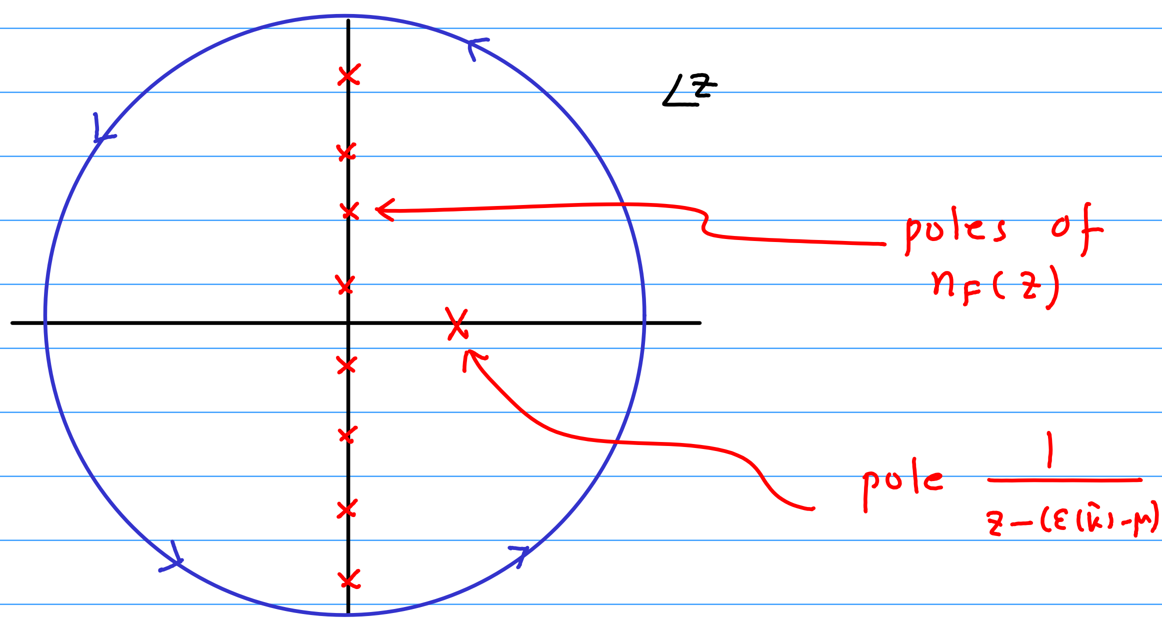
The poles of this function occur at

$$e^{\beta z} = -1 \quad \Rightarrow \quad \boxed{z = \frac{(2n+1)\pi}{\beta}\, i = i p_n} \quad (114)$$

The residue is $\boxed{-\frac{1}{\beta}}$ (115) at each of these poles

Now consider the contour integral over z

$$\boxed{\operatorname*{lt}_{\epsilon\to 0}\ \operatorname*{lt}_{R\to\infty} \oint_{|z|=R} \frac{dz}{2\pi i}\, \frac{n_F(z)}{z-(\varepsilon(\vec{k})-\mu)}\, e^{z\epsilon}} \quad (116)$$



Look at the behavior of the integrand at large $|z|$. This is dominated by the exponential factors.

$$\boxed{\frac{e^{\epsilon z}}{e^{\beta z}+1}} \qquad (117)$$

It is easy to see that since $\beta > \epsilon$ this vanishes for $\text{Re}\, z > 0$. For $\text{Re}\, z < 0$ the denominator $\to 1$ but the numerator now vanishes exponentially. So we conclude that the integrand vanishes everywhere on the contour faster than any power law, and thus

$$\boxed{\lim_{\epsilon\to 0} \lim_{R\to\infty} \oint_{|z|=R} \frac{dz}{2\pi i} \frac{n_F(z)\, e^{\epsilon z}}{z-(\varepsilon(\bar{k})-\mu)} = 0} \qquad (118)$$

On the other hand, the residue theorem tells us that the integral must be the sum of the residues. (119)

$$\Rightarrow \boxed{0 = \underbrace{-\frac{1}{\beta}\sum_{ip_n} \frac{e^{ip_n\epsilon}}{ip_n-(\varepsilon(\bar{k})-\mu)}}_{\text{Comes from the poles of } n_F} + \underbrace{n_F(\varepsilon(\bar{k})-\mu)\, e^{\epsilon(\varepsilon(\bar{k})-\mu)}}_{\text{Comes from } \frac{1}{z-(\varepsilon(\bar{k})-\mu)}}}$$

So, the correlator we want is

$$\boxed{\langle c^\dagger(\tau+\epsilon,\bar{k})\, c(\tau,\bar{k})\rangle = \frac{1}{\beta}\sum_{ip_n} \frac{e^{ip_n\epsilon}}{ip_n-(\varepsilon(\bar{k})-\mu)} = n_F(\varepsilon(\bar{k})-\mu)} \qquad (120)$$

Of course, it is not a surprise that the answer is the Fermi-Dirac occupation, but it is nice to see it emerging from the Grassmann formalism.

Suppose we want to consider arbitrary correlators. The best way to do this is to couple $\bar\psi, \psi$ to sources and take derivatives. For the toy problem (46) we define

$$\boxed{Z[j, \bar j] = \int \prod_{i=1}^{n} d\bar\eta_i \, d\eta_i \; e^{-\bar\eta_i h_{ij} \eta_j + \bar j_i \eta_i + \bar\eta_i j_i}} \qquad (121)$$

where $j_i, \bar j_i$ are Grassmann sources

To do the integral, complete squares, using

$$\boxed{\underset{\sim}{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}} \quad (122) \qquad \boxed{\underset{\sim}{j} = \begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_n \end{bmatrix}} \quad (123) \qquad \text{etc}$$

$$\boxed{\begin{aligned} -\underset{\sim}{\bar\eta}^T h \underset{\sim}{\eta} + \underset{\sim}{\bar j}^T \underset{\sim}{\eta} + \underset{\sim}{\bar\eta}^T \underset{\sim}{j} &= -\left[\underset{\sim}{\bar\eta}^T + \underset{\sim}{\bar j}^T h^{-1}\right] h \left[\underset{\sim}{\eta} + h^{-1} \underset{\sim}{j}\right] \\ &\quad + \underset{\sim}{\bar j}^T h^{-1} \underset{\sim}{j} \end{aligned}} \qquad (124)$$

Do the integral

$$\boxed{Z[\underset{\sim}{j}, \underset{\sim}{\bar j}] = Z_0 \, e^{\underset{\sim}{\bar j}^T h^{-1} \underset{\sim}{j}}} \qquad (125)$$

Now to take correlators, notice that

$$\frac{\partial}{\partial \bar{j}_i} e^{\bar{\underset{\sim}{j}}^T \underset{\sim}{\eta} + \bar{\underset{\sim}{\eta}}^T \underset{\sim}{j}} = \eta_i \, e^{\bar{\underset{\sim}{j}}^T \underset{\sim}{\eta} + \bar{\underset{\sim}{\eta}}^T \underset{\sim}{j}} \tag{126}$$

and

$$\frac{\partial}{\partial j_i} e^{\bar{\underset{\sim}{j}}^T \underset{\sim}{\eta} + \bar{\underset{\sim}{\eta}}^T \underset{\sim}{j}} = -\bar{\eta}_i \, e^{\bar{\underset{\sim}{j}}^T \underset{\sim}{\eta} + \bar{\underset{\sim}{\eta}}^T \underset{\sim}{j}} \tag{127}$$

So

$$\langle \eta_l \bar{\eta}_m \rangle = \left\{ \frac{1}{Z[j,\bar{j}]} \int \mathcal{D}\bar{\eta}\, \mathcal{D}\eta \; \eta_l \bar{\eta}_m \, e^{-\bar{\underset{\sim}{\eta}}^T h \underset{\sim}{\eta} + \bar{\underset{\sim}{j}}^T \underset{\sim}{\eta} + \bar{\underset{\sim}{\eta}}^T \underset{\sim}{j}} \right\}_{\underset{\sim}{j} = \bar{\underset{\sim}{j}} = 0} \tag{128}$$

$$\left\{ \frac{-1}{Z[j\bar{j}]} \int \mathcal{D}\bar{\eta}\, \mathcal{D}\eta \; \frac{\partial}{\partial \bar{j}_l} \frac{\partial}{\partial j_m} e^{-\bar{\underset{\sim}{\eta}}^T h \underset{\sim}{\eta} + \bar{\underset{\sim}{j}}^T \underset{\sim}{\eta} + \bar{\underset{\sim}{\eta}}^T \underset{\sim}{j}} \right\}_{\underset{\sim}{j} = \bar{\underset{\sim}{j}} = 0} \tag{129}$$

$$= \left\{ -\frac{1}{Z[j,\bar{j}]} \frac{\partial}{\partial \bar{j}_l} \frac{\partial}{\partial j_m} Z_0 \, e^{\bar{\underset{\sim}{j}}^T h^{-1} \underset{\sim}{j}} \right\}_{\underset{\sim}{j} = \bar{\underset{\sim}{j}} = 0}$$

$$e^{\bar{\underset{\sim}{j}}^T h^{-1} \underset{\sim}{j}} = e^{\bar{j}_r h^{-1}{}_{rs} j_s} \tag{130}$$

each individual term in the exponent is a bilinear and commutes with every other bilinear

The only term we are interested in is

$$\exp\left(\bar{j}_l h^{-1}{}_{lm} j_m\right) \tag{131}$$ with no sum over l, m

$$\boxed{\exp\left(\bar{j}_\ell h^{-1}_{\ell m} j_m\right) = 1 + \bar{j}_\ell h^{-1}_{\ell m} j_m} \quad (132) \qquad \text{(no sum)}$$

$$\boxed{-\frac{\partial}{\partial \bar{j}_\ell}\frac{\partial}{\partial j_m}\left(1 + \bar{j}_\ell h^{-1}_{\ell m} j_m\right) = +h^{-1}_{\ell m}} \quad (133)$$

So $\boxed{\langle \eta_\ell \bar{\eta}_m \rangle = h^{-1}_{\ell m}}$ (134) which matches (70)

Using the fact that $Z[j,\bar{j}]$ is the exponential of a bilinear we can deduce Wick's theorem, one example of which is

$$\boxed{\langle \eta_i \eta_j \bar{\eta}_\ell \bar{\eta}_m \rangle = \langle \eta_i \bar{\eta}_m \rangle \langle \eta_j \bar{\eta}_\ell \rangle - \langle \eta_i \bar{\eta}_\ell \rangle \langle \eta_j \bar{\eta}_m \rangle} \quad (135)$$