Renormalization Group: What and Why

Throughout physics we make physically motivated models which contain only the phenomenon of interest, and throw away unimportant details. For example, we often ignore air resistance when dealing with a pendulum, because it is small compared to other forces in the problem. This can be dealt with perturbatively.

We also ignore the fact that ultimately the mass of the bob comes from quarks and gluons. This is because the energy scale at which QCD becomes important is much higher than the energy scale of the pendulum.

RG is the process of "integrating out" high-energy (usually also short-distance) degrees of freedom, and examining their impact on the low-energy (usually long-wavelength) physics.

This results in a low-energy effective Hamiltonian, or low-energy effective action. In many-body systems there are excitations in a whole continuum of energies. In this case one "integrates out" excitations above a cutoff, and the parameters of the effective Hamiltonian depend on this cutoff.

Consider some generic Hilbert space which I will assume discrete for simplicity. The Hamiltonian can be written as a matrix, whose precise form depends on the basis. Most often $\hat{\mathcal{H}}$ can be written as

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V} \quad (1)$$

where $\hat{\mathcal{H}}_0$ is exactly soluble, while $\hat{\mathcal{H}}$ is not. For example, an anharmonic oscillator

$$\hat{\mathcal{H}}_{osc} = \underbrace{\frac{p^2}{2m} + \frac{1}{2}kx^2}_{\hat{\mathcal{H}}_0} + \underbrace{\frac{\lambda}{4}x^4}_{V} \quad (2)$$

Diagonalize $\hat{\mathcal{H}}_0$ and order the eigenstates according to the energy (of $\hat{\mathcal{H}}_0$). For $\hat{\mathcal{H}}_{osc}$ we know

$$\hat{\mathcal{H}}_0 |n\rangle = \hbar\omega_0 \left(n + \tfrac{1}{2}\right)|n\rangle \quad (3)$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad (4)$$

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \quad (5)$$

where $|0\rangle$ is the ground state of $\hat{\mathcal{H}}_0$ and the ladder operators $a, a^\dagger$ are

$$a = \frac{\sqrt{k}\,x + ip/\sqrt{m}}{\sqrt{2\hbar\omega_0}}, \quad a^\dagger = \frac{\sqrt{k}\,x - ip/\sqrt{m}}{\sqrt{2\hbar\omega_0}} \quad (6)$$

$$[a, a^\dagger] = 1 \quad (7)$$

In this basis V is generically not diagonal (though it may have diagonal elements) For the oscillator

$$\hat{\mathcal{H}}_0 = \omega_0 \left(a^\dagger a + 1/2\right) \quad (8)$$

$\hbar = 1$ (9) hereafter

$$x = \sqrt{\frac{\omega_0}{2k}} \left(a + a^\dagger\right) \quad (10)$$

So

$$\hat{V} = \frac{\lambda}{4} \left(\frac{\omega_0}{2k}\right)^2 \left(a + a^\dagger\right)^4 \quad (11)$$

We are interested in the low-energy effective Hamiltonian. Let us truncate the Hilbert space to $n_{max} = N$. Then $\hat{\mathcal{H}}$ is an $N \times N$ matrix $|H|$

An extremely useful object is a Green's f^n, which is an operator in the Hilbert space

$$\left(i\frac{\partial}{\partial t} - \hat{\mathcal{H}}\right) \hat{G}(t,t') = \underline{\mathbb{1}}\, \delta(t-t') \quad (12)$$

where the RHS is the notation $\underline{\mathbb{1}}$ stands for the identity operator. In our basis, the Green's f^n is the matrix G

$$\left(i\partial_t - |H|\right) G(t,t') = \underline{\mathbb{1}}\, \delta(t-t') \quad (13)$$

Fourier transform in time. Since $|H|$ is time-independent, by time-translation invariance

$$G(t,t') \equiv G(t-t') \quad (14)$$

$$G(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega(t-t')} \, G(\omega) \qquad (15)$$

$$G(\omega) = \int_{-\infty}^{\infty} d\tau \, e^{i\omega\tau} \, G(\tau)$$

(16)

$$\text{So} \quad \int_{-\infty}^{\infty} d\tau \, e^{i\omega\tau} \left(i\partial_\tau - H \right) G(\tau) = \mathbb{1} \int_{-\infty}^{\infty} d\tau \, e^{i\omega\tau} \delta(\tau) = \mathbb{1}$$

Integrate by parts on the 1st term and drop boundary terms (why is one allowed to do this?)

$$(\omega - H)\, G(\omega) = \mathbb{1} \qquad (17)$$

$$\Rightarrow \quad G(\omega) = (\omega - H)^{-1} \qquad (18)$$

Suppose we could diagonalize H exactly (we can't)

$$H |\alpha\rangle = \varepsilon_\alpha |\alpha\rangle \qquad (19)$$

The we can write H in its spectral representation

$$H = \sum_\alpha |\alpha\rangle \, \varepsilon_\alpha \, \langle\alpha| \qquad (20)$$

and thus

$$G = \sum_\alpha |\alpha\rangle \, \frac{1}{\omega - \varepsilon_\alpha} \, \langle\alpha| \qquad (21)$$

The key point is that the poles of G are eigenvalues of H.

Now go back to our H which we cannot exactly solve

$$H = \begin{bmatrix} \varepsilon_0 + V_{00} & & & \\ & \varepsilon_1 + V_{11} & & \\ & & \varepsilon_2 + V_{22} & \\ & & & \ddots \\ & & & & \varepsilon_N + V_{NN} \end{bmatrix} + V' \qquad (22)$$

where V' is the same as V with the diagonal elements subtracted.

Now we will "integrate out" the highest energy state N. Of course, we cannot do this exactly, but we can do it well enough to get a good approximation for the low-energy effective H.

To do this we look to the Green's f^n G

Let us generically consider

$$H = \begin{bmatrix} H_{ll} & H_{lh} \\ H_{hl} & H_{hh} \end{bmatrix} \qquad (23)$$

$$\omega - H = \begin{bmatrix} \omega - H_{ll} & -H_{lh} \\ -H_{hl} & \omega - H_{hh} \end{bmatrix} \qquad (24)$$

$$G = (\omega - H)^{-1} = \begin{bmatrix} G_{ll} & G_{lh} \\ G_{hl} & G_{hh} \end{bmatrix} \qquad (25)$$

$H_{hl} = H_{lh}^{+}$

$G_{hl} = G_{lh}^{+}$

l = low energy

h = high energy

These two subspaces could have different dimensions!

Now $(\omega - H)\, G = \mathbb{1}$

$$\Rightarrow \begin{bmatrix} \omega - \mathbb{H}_{ll} & -\mathbb{H}_{lh} \\ -\mathbb{H}_{hl} & \omega - \mathbb{H}_{hh} \end{bmatrix} \begin{bmatrix} \mathbb{G}_{ll}(\omega) & \mathbb{G}_{lh}(\omega) \\ \mathbb{G}_{hl}(\omega) & \mathbb{G}_{hh}(\omega) \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{ll} & 0 \\ 0 & \mathbb{1}_{hh} \end{bmatrix} \quad (26)$$

or

$$(\omega - \mathbb{H}_{ll})\,\mathbb{G}_{ll} - \mathbb{H}_{lh}\,\mathbb{G}_{hl} = \mathbb{1}_{ll} \quad (27)$$

$$(\omega - \mathbb{H}_{ll})\,\mathbb{G}_{lh} - \mathbb{H}_{lh}\,\mathbb{G}_{hh} = 0 \quad (28)$$

$$-\mathbb{H}_{hl}\,\mathbb{G}_{ll} + (\omega - \mathbb{H}_{hh})\,\mathbb{G}_{hl} = 0 \quad (29)$$

$$-\mathbb{H}_{hl}\,\mathbb{G}_{lh} + (\omega - \mathbb{H}_{hh})\,\mathbb{G}_{hh} = \mathbb{1}_{hh} \quad (30)$$

From (29)

$$\mathbb{G}_{hl} = (\omega - \mathbb{H}_{hh})^{-1}\,\mathbb{H}_{hl}\,\mathbb{G}_{ll}$$

Plug this into (27)

$$\left[\omega - \mathbb{H}_{ll} - \mathbb{H}_{lh}(\omega - \mathbb{H}_{hh})^{-1}\mathbb{H}_{hl}\right]\mathbb{G}_{ll} = \mathbb{1} \quad (31)$$

So far no approximation has been made. Now, we want low-energy poles of $\mathbb{G}_{ll}$, which means that ω is small near the desired poles. So, we are justified in expanding

$$(\omega - \mathbb{H}_{hh})^{-1} = -\mathbb{H}_{hh}^{-1} - \omega\,\mathbb{H}_{hh}^{-2} + \cdots \quad (32)$$

Let us keep only these terms

$$\Rightarrow \quad G_{ll}(\omega) \simeq \left[\omega \left(\mathbb{1} - H_{lh} H_{hh}^{-2} H_{hl} \right) - \left(H_{ll} - H_{lh} H_{hh}^{-1} H_{hl} \right) \right]^{-1} \tag{33}$$

This is equivalent to a generalized eigenvalue problem in the l-subspace. The 2nd part is the renormalized effective Hamiltonian

$$H_{ll,\,eff} = H_{ll} - H_{lh} H_{hh}^{-1} H_{hl} \tag{34}$$

and the 1st part is "wave-function renormalization"

We have integrated out the h-subspace and obtained a renormalized Hamiltonian in the l-subspace.

As long as we keep only terms of ω^0 and ω^1 the renormalized problem is also Hamiltonian. If, however, we keep higher powers of ω, we can no longer write G_{ll} as the solution to a 1st-order (in t) differential eqn (13) but will involve higher time-derivatives

These higher-order derivatives signify nonlocal couplings in time, and their proper treatment need an action formulation leading to a path integral.

Doing this procedure successively to the highest energy state remaining leads to

a discrete flow of $\mathcal{H}_{\ell\ell}$. Often (but not always) the renormalized hamiltonian can be written exactly like the original Hamiltonian, except that the parameters (such as m, k and λ in (2)) change at each iteration of the RG procedure.

Imagine that we have a complete set of parameters $\{g_i\}$ characterizing $\mathcal{H}_{\ell\ell}$. Then each RG step can be written as a mapping

$$g_i^{(n+1)} = f_i(\{g_j^{(n)}\}) \quad (35)$$

where $\{g_j^{(n)}\}$ are the parameters defining $\mathcal{H}_{\ell\ell}$ at step n of the RG.

Particular importance must be paid to fixed points of the RG, where the new parameters are identical to the old. If $\{g_i^*\}$ define the fixed point

$$g_i^* = f_i(\{g_j^*\}) \quad (36)$$

To see the significance of fixed points we will need to learn more about many-body systems and length scales.

For the moment, let us assume, as is usually the case, that energy and length scales are inversely related: High energies correspond to short length scales and low energies to long length scales

A fixed point corresponds to scale-invariant physics!!

Now assume that the system is near a fixed point

$$g_i^{(0)} = g_i^* + \delta g_i^{(0)} \quad (37)$$

For small enough $\delta g_i^{(0)}$ one can expand

$$g_i^{(1)} = f_i(g_i^* + \delta g_i) = f_i(g_i^*) + \delta g_j \left.\frac{\partial f_i}{\partial g_j}\right|^* + \cdots \quad (38)$$

or if

$$g_i^{(1)} = g_i^* + \delta g_i^{(1)} \quad (39)$$

$$\delta g_i^{(1)} \approx M_{ij}(*)\, \delta g_i^{(0)} \quad (40)$$

Near a fixed pt any mapping is approximately linear.

The eigenvalues of M play a crucial role.

If an eigenvalue of M is greater than 1 the corresponding δg will grow under RG. Such a δg is called relevant

If an eigenvalue is less than 1, the corresponding δg is irrelevant, because

it shrinks under each RG operation. If the eigenvalue is 1, the Sg is said to be marginal.

The number of relevant eigenvalues tells us how many parameters have to be tuned in the original problem to achieve the scale-invariant physics represented by the fixed pt.

The magnitude of the relevant eigenvalue tells us something about how physical quantities diverge (or vanish) as one approaches the scale-invariant pt.

Scale invariance occurs naturally at continuous phase transitions (those with no Latent heat), and also in time-dependent phenomena such as turbulence.

RG is a natural tool for the study of such systems.