Scaling near the critical point

We have defined the following critical exponents

$$\boxed{C_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_{V,N} = C_0 |T-T_c|^{-\alpha}} \quad ①$$

$$\boxed{\langle\phi\rangle(T) = -\frac{1}{N_0}\frac{\partial F}{\partial h}\bigg|_{h=0} = \phi_0 (T_c - T)^{\beta}} \quad ② \quad T < T_c$$

N_0 = # of sites

For an external field h

$$\boxed{\chi(T) = \frac{\partial\langle\phi\rangle}{\partial h}\bigg|_{h=0} = -\frac{\partial^2 F}{\partial h^2}\bigg|_{h=0} = \chi_0 |T-T_c|^{-\gamma}} \quad ③$$

$$\boxed{\langle\phi(\bar{x})\phi(0)\rangle_c = \frac{e^{-|\bar{x}|/\xi(T)}}{|\bar{x}|^{d-2+\eta}}} \quad ④$$

$$\boxed{\xi(T) = a|T-T_c|^{-\nu}} \quad ⑤$$

a = "lattice spacing" = length cutoff

At $T = T_c$

$$\boxed{\langle\phi(h)\rangle = \phi_0 h^{\frac{1}{\delta}}} \quad ⑥$$

It turns out that these exponents are not all independent. Relations between them can be derived by a single fundamental scaling assumption.

To motivate the scaling assumption consider a free energy $F(\tau, h)$ where $\boxed{\tau = T - T_c}$ (7) and h is the external field coupling to ϕ. In the Ising model, for example

$$\boxed{S(\tau, h) = \underbrace{\int d^d x \left\{ \frac{1}{2} (\bar{\nabla} \phi)^2 + \frac{\tau}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right.}_{S_0} \left. - h\phi \right\}} \quad (8)$$

$$\boxed{Z = \int \mathcal{D}\phi(\bar{x})\, e^{-S/T} = e^{-F(\tau, h)/T}} \quad (9)$$

Let us integrate out the short-distance degrees of freedom between the lattice scale a and an intermediate scale ab

(10) $\boxed{b > 1}$ is a dimensionless scaling factor.

To be explicit, divide $\phi(\bar{x})$ into a slow part $\phi_<(\bar{x})$ and a fast part $\phi_>(\bar{x})$

$$\boxed{\begin{aligned} \phi_<(\bar{x}) &= \int_0^{2\pi/ab} \frac{d^d k}{(2\pi)^d}\, e^{i\bar{k}\cdot\bar{x}}\, \tilde{\phi}_<(\bar{k}) \\ \phi_>(\bar{x}) &= \int_{2\pi/ab}^{2\pi/a} \frac{d^d k}{(2\pi)^d}\, e^{i\bar{k}\cdot\bar{x}}\, \tilde{\phi}_>(\bar{k}) \end{aligned}} \quad (11)$$

$$\boxed{Z = \int \mathcal{D}\phi(\bar{x})\, e^{-S/T} = \int \mathcal{D}\phi_<(\bar{x}) \int \mathcal{D}\phi_>(\bar{x})\, e^{-S/T}} \quad (12)$$

Under this procedure, we know that the couplings in S will change. Also, only the relevant couplings, τ and h, need be kept. So, under this rescaling

$$\boxed{\tau \to \tau b^x \qquad h \to h b^y} \quad (13)$$

(14) $\boxed{x, y > 0}$ come from the eigenvalues of the linearized RG eq^n near the fixed point.

$$\boxed{Z = \int \mathcal{D}\phi_<(\bar{x})\, e^{-S'(\tau b^x, h b^y)/T} \; e^{-\frac{F_{reg}(b)}{T}}} \quad (15)$$

The last factor is an additive constant (in the sense that it does not contain $\phi_<(\bar{x})$) and is called the regular part of the Free energy. It cannot contain any singularities, because all singularities near T_c come from slow fluctuations.

Let the original number of sites be N_0. After rescaling it becomes N_0/b^d.

Let us further define an intensive quantity

$$\boxed{f = \text{Free energy per site}} \quad (16)$$

Since

$$\boxed{F_{sing}(\tau, h) = F_{sing}(\tau b^x, h b^y)} \quad (17)$$

we get the fundamental scaling relation

$$\boxed{f_{sing}(\tau, h) = f_{sing}(\tau b^x, h b^y)/b^d} \quad (18)$$

Often this is recast as follows. Let us choose b such that

$$\boxed{\tau \bar{b}^x = 1 \implies \bar{b} = \tau^{-\frac{1}{x}}} \quad (19)$$

$$\implies \boxed{f_{sing}(\tau, h) = \tau^{d/x}\, \Psi\left(\frac{h}{\tau^{y/x}}\right)} \quad (20)$$

where

$$\boxed{\Psi(z) = f_{sing}(1, z)} \quad (21)$$

To be more precise, Ψ can have different functional forms above (Ψ_+) and below (Ψ_-) T_c

Start with below T_c, at $h=0$

$$\boxed{f_{sing}(\tau, 0) = \tau^{d/x}\, \Psi_-(0)} \quad (22) \quad T < T_c$$

Take two derivatives w.r.t. τ to obtain

$$\boxed{C_V \sim \tau^{\frac{d}{x}-2}\, \Psi_-(0)} \quad (23)$$

$$\implies \boxed{\alpha = 2 - \frac{d}{x}} \quad (24)$$

Now the magnetization is

$$\boxed{\langle\phi\rangle = -\left.\frac{\partial f}{\partial h}\right|_{h=0} = -\tau^{\frac{d}{x}}\frac{\partial}{\partial h}\Psi_-\left(\frac{h}{\tau^{y/x}}\right)\Bigg|_{h=0}} \quad (25)$$

$$\boxed{\langle\phi\rangle = -\tau^{\frac{d-y}{x}}\,\Psi_-'(0)} \quad (26)$$

where the prime is differentiation w.r.t. to the argument.

$$\Rightarrow \quad \boxed{\beta = \frac{d-y}{x}} \quad (27)$$

Incidentally, applying this to $T>T_c$ shows that

$$\boxed{\Psi_+'(0) = 0} \quad (28)$$

Now consider the susceptibility

$$\boxed{\chi = -\left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} = -\tau^{\frac{d-2y}{x}}\,\Psi_\pm''(0)} \quad (29)$$

$$\Rightarrow \quad \boxed{\gamma = \frac{2y-d}{x}} \quad (30)$$

There is a deep connection between the susceptibility and the correlation function

Recall

$$\boxed{\langle\phi(\tau,h)\rangle = \frac{1}{Z}\int\mathcal{D}\phi(\bar{y})\, e^{-\frac{S_0[\phi]}{T} + \frac{h}{T}\int d^dy\,\phi(\bar{y})}\,\phi(\bar{x})} \quad (31)$$

$$\boxed{\begin{aligned} \chi(\tau,h) &= \frac{\partial}{\partial h}\langle \phi(\tau,h)\rangle\Big|_{h=0} \qquad (32)\\ &= -\frac{1}{Z_1^2}\frac{\partial Z_1}{\partial h}\int \mathcal{D}\phi\, e^{-\frac{S_0'[\phi]}{T}+\frac{h}{T}\int d^dy\,\phi(\bar{y})}\phi(\bar{x}) \\ &\quad + \frac{1}{Z_1}\int \mathcal{D}\phi\, e^{-\frac{S_0[\phi]}{T}+\frac{h}{T}\int d^dy\,\phi(\bar{y})}\frac{1}{T}\int d^dy\,\phi(\bar{y})\phi(\bar{x}) \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{1}{Z_1}\frac{\partial Z_1}{\partial h}\Big|_{h=0} &= \frac{1}{Z}\frac{1}{T}\int \mathcal{D}\phi(\bar{y})\, e^{-\frac{S_0[\phi]}{T}}\int d^dx\,\phi(\bar{x}) \qquad (33)\\ &= \frac{N_0}{T}\langle\phi\rangle \end{aligned}}$$

$$\boxed{\chi = \frac{1}{T}\int d^dy\left\{\langle\phi(\bar{x})\phi(\bar{y})\rangle - \langle\phi\rangle^2\right\} = \frac{1}{T}\int d^dy\,\langle\phi(\bar{x})\phi(\bar{y})\rangle_c} \qquad (34)$$

Now we know that the connected correlator goes as

$$\frac{e^{-|\bar{x}-\bar{y}|/\xi}}{|\bar{x}-\bar{y}|^{d-2+\eta}}$$

So we want $\boxed{\frac{1}{T}\int d^dy\,\frac{e^{-|\bar{y}|/\xi}}{y^{d-2+\eta}}}$ (35)

rescale $\boxed{y \to y/\xi = \zeta}$ (36)

$$\boxed{\chi = \frac{1}{T}\xi^{2-\eta}\int d^d\zeta \; \frac{e^{-\zeta}}{\zeta^{d-2+\eta}}} \quad (37)$$

The integral has no dependence on τ, h

So, near T_c

$$\boxed{\chi(\tau) \sim \tau^{-\gamma} = \tau^{-\nu(2-\eta)}} \quad (38)$$

$$\Rightarrow \boxed{\gamma = \nu(2-\eta)} \quad (39)$$

Now combine

$$\boxed{\alpha = 2 - \frac{d}{x}} \quad (24) \qquad \boxed{\beta = \frac{d-y}{x}} \quad (27) \qquad \boxed{\gamma = \frac{2y-d}{x}} \quad (30)$$

$$\boxed{\frac{d}{x} = 2-\alpha} \quad (40) \quad \Rightarrow \quad \beta = 2-\alpha - \frac{y}{x}$$

$$\boxed{\frac{y}{x} = 2-\alpha-\beta} \quad (41)$$

So $$\boxed{\gamma = 2(2-\alpha-\beta) - 2 + \alpha = 2 - \alpha - 2\beta} \quad (42)$$

Now go back to (22) $T < T_c$

$$f_{sing} = \tau^{d/x}\, \Psi_-\left(\frac{h}{\tau^{y/x}}\right)$$

$$\boxed{\langle \phi \rangle = \tau^{\frac{d-y}{x}}\, \Psi_-'\left(\frac{h}{\tau^{y/x}}\right)} \quad (43)$$

Now take the limit $\tau \to 0$. We know, by definition, that in this limit

$$\langle \phi \rangle \sim h^{1/\delta}$$

This means that as $\zeta = \frac{h}{\tau^{y/x}} \to \infty$

$$\boxed{\Psi_-'(\zeta) \sim \zeta^{1/\delta}} \quad (44)$$

$$\boxed{\langle \phi \rangle \sim \tau^{\frac{d-y}{x}} \frac{h^{\frac{1}{\delta}}}{\tau^{y/\delta x}}} \quad (45)$$

This should be finite as $\tau \to 0$ so the powers of τ should cancel!

$$\Rightarrow \quad \boxed{\frac{d-y}{x} = \frac{y}{\delta x}} \quad (46)$$

$$\text{or} \quad \boxed{\frac{y}{d} = \frac{\delta}{\delta+1}} \quad (47)$$

$$\text{Now} \quad \frac{y}{x} = \frac{y/d}{x/d} = \frac{\frac{\delta}{\delta+1}}{\frac{1}{2-\alpha}} = 2-\alpha-\beta$$

$$\Rightarrow \quad (2-\alpha)\delta = (2-\alpha-\beta)(\delta+1)$$

$$\Rightarrow \quad 0 = 2-\alpha-\beta(\delta+1)$$

$$\boxed{\alpha+\beta(\delta+1) = 2} \quad (48)$$

The scaling relations between the exponents (39), (42), (48) do not depend on d, and always hold.

The final relation has the dimension d in it and holds only below the "upper critical dimension" (4 for the Ising Model)

This argument says that the only relevant length scale for f_{sing} is the correlation length. Since f_{sing} has units of energy/volume

$$\boxed{f_{sing}(\tau) = \frac{f_0}{\xi^d} = f_0\, \tau^{\nu d}} \quad (49)$$

$$\Rightarrow \quad \boxed{x = \frac{1}{\nu}} \quad (50) \qquad \text{and} \qquad \boxed{\alpha = 2 - \nu d} \quad (51)$$

(51) is called hyperscaling.

As an example, for the 2D Ising Model

$$\boxed{\alpha = 0 \qquad \beta = \frac{1}{8} \qquad \gamma = \frac{7}{4}} \quad (52)$$

$$\boxed{\nu = \frac{2-\alpha}{d} = \frac{2}{d} = 1} \quad (53)$$

$$\gamma = \nu(2-\eta) \Rightarrow \eta = 2 - \frac{\gamma}{\nu} = 2 - \frac{7}{4} = \frac{1}{4} \qquad \boxed{\eta = \frac{1}{4}} \quad (54)$$

$$\alpha + \beta(\delta+1) = 2 \Rightarrow \delta + 1 = \frac{2}{\beta} = 16 \qquad \boxed{\delta = 15} \quad (55)$$