Symmetry, the Ginzburg criterion and the upper critical dimension

Let us first talk about internal symmetries.

We have considered the Ising model, which at the microscopic level, has $S_i = \pm 1$. This arises from the interplay of many solid-state effects: Crystal symmetry, spin-orbit coupling, electron-electron interactions etc.

There are 3 broad possibilities in a real material, where there are unpaired spins.

(i) There is no spin-orbit coupling. This happens in materials composed of elements with low atomic number. Then the symmetry of the spins, and the interaction between spins, is $SU(2)$

(ii) Spin-orbit and crystal fields select a preferred axis (the easy axis). This comes from a term of the form $-D S_z^2$ $(D>0)$. This leads to the Ising symmetry we have studied, formally denoted by the group $\mathbb{Z}_2$ composed of 1 and -1 under multiplication. If all spins are multiplied by -1, the energy does not change

(iii) Spin-orbit and crystal fields select an easy plane via a term $+D S_z^2$ $(D>0)$. Then the spin prefers to lie in the XY plane, and the symmetry is XY-like, corresponding to the group $U(1)$ or $O(2)$.

Let us look at the coarse-grained action for each of these symmetries and their generalizations.

Ising: If the slow fluctuations of the order parameter are denoted by $\phi(\bar{x})$ then $\boxed{\phi(\bar{x}) \to -\phi(\bar{x})}$ (1) (for all $\bar{x}$ simultaneously, a global symmetry) should leave the action unchanged. So only even powers of ϕ are allowed

$$\boxed{S = \frac{1}{2}(\bar{\nabla}\phi)^2 + \frac{r}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \cdots} \quad (2)$$

A natural generalization of the Ising, or Z_2 symmetry, is the Z_n group, composed of all the n^{th} roots of unity under multiplication

$$\boxed{Z_3 : \{1, \omega, \omega^2\} \qquad \omega = e^{2\pi i/3}} \quad (3)$$

Clearly, in this case, the action must be invariant under the global transformation

$$\boxed{\phi \to \phi e^{i2\pi/n}} \quad (4)$$

One way to encode this symmetry is to integrate over all complex number ϕ, but put a constraint into the path integral

$$\boxed{Z = \int \mathcal{D}\phi^* \mathcal{D}\phi \left\{ \prod_{\vec{x}} \delta[\phi^n(\vec{x})-1] \right\} e^{-S/T}} \quad (5)$$

Now "soften" the constraint, replacing it by

$$\boxed{\prod_{\vec{x}} \delta[\phi^n(\vec{x})-1] \approx e^{-\frac{1}{2\sigma^2}\int d^d x \, [\phi^n(\vec{x})-1][\phi^{*n}(\vec{x})-1]}} \quad (6)$$

So

$$\boxed{S = \int d^d x \left\{ \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + r\phi^*\phi + \lambda(\phi^*\phi)^2 + \lambda_n \phi^n + \lambda_n \phi^{*n} + \cdots \right\}} \quad (7)$$

These additional terms are relevant for $n=3$, and marginal (does not change under RG) for $n=4$.

XY: The spin can be characterized by an angle $\varphi(\vec{x})$, and the global symmetry is

$$\boxed{\varphi(\vec{x}) \to \varphi(\vec{x}) + \varphi_0} \quad (8) \qquad (\text{all } \vec{x})$$

On the lattice one can write

$$\boxed{\mathcal{E}(\{\varphi\}) = -J \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j)} \quad (9)$$

In the coarse grained theory there are many ways to encode $U(1) = O(2)$

a) A single complex field $\Phi(\bar{x})$

$$\boxed{\Phi(\bar{x}) \rightarrow \Phi(\bar{x})\, e^{i\varphi_0}} \quad (10)$$

is the global symmetry

$$\boxed{S = \int d^dx \left\{ \vec{\nabla}\Phi^* \cdot \vec{\nabla}\Phi + r\, \Phi^*\Phi + \lambda (\Phi^*\Phi)^2 + \cdots \right\}} \quad (11)$$

b) A pair of real fields $\boxed{\Phi(\bar{x}) = \phi_1(\bar{x}) + i\phi_2(\bar{x})}$ (12)

$$\boxed{\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{bmatrix} \cos\varphi_0 & \sin\varphi_0 \\ -\sin\varphi_0 & \cos\varphi_0 \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}} \quad (13)$$

is the $O(2)$ global

$$\boxed{S = \int d^dx \left[(\vec{\nabla}\phi_1)^2 + (\vec{\nabla}\phi_2)^2 + r(\phi_1^2 + \phi_2^2) + \lambda(\phi_1^2 + \phi_2^2)^2 + \cdots \right]} \quad (14)$$

This is easily generalized to $O(N)$

$$\boxed{S = \int d^dx \left[\vec{\nabla}\phi_\alpha \cdot \vec{\nabla}\phi_\alpha + r\, \phi_\alpha\phi_\alpha + \lambda(\phi_\alpha\phi_\alpha)^2 + \cdots \right]} \quad (15)$$

where a repeated index (such as α) is to be summed between 1 and N.

Note that ϕ_α are real, and $O(N)$ is non-Abelian.

Heisenberg or $SU(2)$: This is the case with no spin-orbit coupling. The order parameter is a 2-component complex spinor $\boxed{\begin{pmatrix} \psi_\uparrow(\bar{x}) \\ \psi_\downarrow(\bar{x}) \end{pmatrix}}$ (16)

and the global symmetry operation is

$$\boxed{\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \rightarrow e^{i\frac{\theta}{2}\vec{\sigma}\cdot\hat{n}} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}} \quad (17)$$

where $\vec{\sigma}$ are the Pauli matrices.
This can be generalized to an N-component complex spinor $\psi_a(\bar{x})$, with the global symmetry being $SU(N)$. A very convenient fact is that $SU(2)$ and $O(3)$ are identical except for global topology.

Finally, let us talk about superconductors, which was the problem for which Ginzburg-Landau theory was originally invented.

The order parameter is the pair wave function, which is complex, $\boxed{\Psi(\bar{x})}$ (18). Since the superconductors known then were s-wave spin-singlet superconductors, Ψ has no internal degrees of freedom other that its phase.

$$\boxed{\Psi(\bar{x}) = |\Psi(\bar{x})|\, e^{i\varphi(\bar{x})}} \quad (19)$$

At the moment this looks identical to the XY case. However, there is a crucial difference.

An XY spin is neutral but a Cooper pair is charged. Any object with a charge has to respond to electromagnetic fields. The most natural couplings of particles to the EM field is via the scalar and vector potentials $(\Phi, \vec{A})$. These are not physical themselves because a gauge transformation

$$\boxed{\begin{aligned} \Phi(\bar{x},t) &\mapsto \Phi(\bar{x},t) + \frac{\partial \chi(\bar{x},t)}{\partial t} \\ \vec{A}(\bar{x},t) &\mapsto \vec{A}(\bar{x},t) - \vec{\nabla}\chi(\bar{x},t) \end{aligned}} \quad (20)$$

leads to the same physical $\vec{E}, \bar{B}$. Under this gauge transformation the wave function responds with a local phase transformation

$$\boxed{\Psi(\bar{x},t) \mapsto \Psi(\bar{x},t)\, e^{\frac{ie}{\hbar}\chi(\bar{x},t)}} \quad (21)$$

So the action must be invariant under such transformations. Restricting to equilibrium stat mech (no t-dependence) the coarse-grained action must be

$$\boxed{S = \int d^dx \left\{ \left[\left(\vec{\nabla} + \frac{ie}{\hbar}\vec{A}\right)\Psi^*\right] \cdot \left[\left(\vec{\nabla} - \frac{ie}{\hbar}\vec{A}\right)\Psi\right] + r\,\Psi^*\Psi + \lambda\left(\Psi^*\Psi\right)^2 + \cdots \right\}} \quad (22)$$

At the moment $\vec{A}$ is an external field and does not have statistical fluctuations.

The Ginzburg Criterion

Back to the simple Ising Model.

The Ginzburg criterion tells us how close to the critical point one can be and still carry out perturbation theory. Start with our coarse-grained Ising Model

$$\boxed{S = \int \frac{d^dx}{a^d} \left\{ J\frac{a^2}{2} (\vec{\nabla}\phi)^2 + Jr\,\phi^2 + \frac{J\lambda}{4}\phi^4 \right\}} \quad (23)$$

J is an energy, a is the lattice spacing and ϕ is dimensionless. r is the dimensionless distance from criticality

$$\boxed{r = \frac{T - T_c}{T_c}} \quad (24)$$

Let us rescale $\vec{x}$ such that

$$\boxed{\vec{y} = \frac{\vec{x}\sqrt{r}}{a}} \quad (25) \quad \Rightarrow \quad \boxed{\vec{\nabla}_{\vec{x}} = \frac{\sqrt{r}}{a}\vec{\nabla}_{\vec{y}}} \quad (26)$$

$$S = J\frac{1}{r^{d/2}} \int d^d\vec{y} \left\{ a^2 \frac{r}{2a^2} (\vec{\nabla}_{\vec{y}}\phi)^2 + \frac{r}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \right\}$$

$$\boxed{S = J\,r^{1-\frac{d}{2}} \int d^d\vec{y} \left\{ \frac{(\vec{\nabla}_{\vec{y}}\phi)^2}{2} + \frac{\phi^2}{2} + \frac{\lambda}{4r}\phi^4 \right\}} \quad (27)$$

Now we rescale the field in order to absorb the $r^{1-d/2}$ factor in front

$$\boxed{r^{\frac{2-d}{4}}\phi(y) = \tilde{\phi}(y) \Rightarrow \phi^4 = \tilde{\phi}^4 r^{d-2}} \quad (28)$$

$$\boxed{S = J\int d^dy\left\{\frac{(\vec{\nabla}\tilde{\phi})^2}{2} + \frac{r}{2}\tilde{\phi}^2 + \frac{\lambda}{4r} r^{1-\frac{d}{2}} r^{d-2}\tilde{\phi}^4\right\}}$$

$\Rightarrow$ The $\lambda\phi^4$ term is of order

$$\boxed{r^{1-\frac{d}{2}}\frac{\lambda}{r} r^{d-2} = \lambda r^{\frac{d}{2}-2}} \quad (30)$$

As one gets closer to the critical pt $r \to 0$ the λ term becomes less important if $d > 4$. This means perturbation theory in λ should be valid even near the critical point, which means mean-field theory should be valid.

However if $d < 4$, no matter how small λ is, close enough to the critical point $\lambda r^{\frac{d}{2}-2}$ will become large, and perturbation theory will become invalid.

$d_u = 4$ is known as the upper critical dimension.

Let us compute something physical to check that this hand-waving argument is correct.

The simplest object is the free energy itself. To set up the calculation we will demonstrate a very useful result.

Assume that the action has a piece S_0 quadratic in the fields and another piece V which is not quadratic

For the Ising Model

$$\boxed{S_0 = \int d^d x \left\{ \frac{1}{2}(\bar{\nabla}\phi)^2 + \frac{r}{2}\phi^2 \right\}} \quad (31)$$

$$\boxed{V = \int d^d x \; \frac{\lambda}{4}\phi^4(\bar{x})} \quad (32)$$

Let us denote $\langle \mathcal{O} \rangle_0$ to be the average of some quantity $\mathcal{O}$ in the ensemble given by S_0.

$$\boxed{\langle \mathcal{O} \rangle_0 = \frac{\int \mathcal{D}\phi(\bar{x})\, e^{-\frac{S_0[\phi]}{T}}\, \mathcal{O}}{\int \mathcal{D}\phi(\bar{x})\, e^{-S_0[\phi]/T}}} \quad (33)$$

The advantage of S_0 is that it leads (after diagonalizing the Kernel) to a product of Gaussian Integrals, and all correlators obey Wick's theorem

Let us recall Wick's theorem

$$\boxed{\langle \phi(\bar{x}_1)\,\phi(\bar{x}_2)\cdots\phi(\bar{x}_n)\rangle_0 = \sum_{\substack{\text{all}\\ \text{pairs}}} \langle \phi(\bar{x}_1)\,\phi(\bar{y}_1)\rangle_0 \langle \phi(\bar{x}_2)\,\phi(\bar{y}_2)\rangle_0 \cdots} \quad (34)$$

Divide the n ϕ's into $\frac{n}{2}$ pairs. If n is odd the correlator is zero. There are

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!}$$ ways of choosing the 1st pair

$$\binom{n-2}{2} = \frac{(n-2)!}{2!\,(n-4)!}$$ ways of choosing the next pair etc. So the number of pairs seems to be

$$\frac{n!}{(2!)^{n/2}}$$

However, different orders of selecting the pairs are identical, so one must divide by $\left(\frac{n}{2}\right)!$ = # of ways of arranging the pairs

$$\boxed{\#\text{ of different pairings} = \frac{n!}{\left(\frac{n}{2}\right)!}\, 2^{-\frac{n}{2}}} \quad (35)$$

Now we need to introduce the related, but slightly different, idea of a cumulant

Consider O_1 O_2 ... O_n which are functions of some random variable

The cumulant

$$\boxed{\langle O_1 O_2 \cdots O_n \rangle_c = \langle O_1 O_2 \cdots O_n \rangle - \sum_{S_k} \langle \prod_{i \in S_1} O_i \rangle \otimes \langle \prod_{i \in S_2} O_i \rangle \cdots \langle \prod_{i \in S_k} O_i \rangle} \quad (36)$$

S_k are all the proper partitions of the set $\{1, 2, \cdots, n\}$, which means they are all the ways of dividing up the set into proper subsets

For example, for $n=3$ the proper partitions are

$\{1\}, \{2,3\}$

$\{1,2\}, \{3\}$

$\{1,3\}, \{2\}$

$\{1\}, \{2\}, \{3\}$

So
$$\boxed{\begin{aligned}\langle O_1 O_2 O_3 \rangle_c &= \langle O_1 O_2 O_3 \rangle - \langle O_1 \rangle \langle O_2 O_3 \rangle \\ &\quad - \langle O_2 \rangle \langle O_1 O_3 \rangle - \langle O_3 \rangle \langle O_1 O_2 \rangle \\ &\quad + 2 \langle O_1 \rangle \langle O_2 \rangle \langle O_3 \rangle\end{aligned}} \quad (37)$$

The cumulant measures the correlations of the product not contained in any subcorrelation

Armed with the cumulant, we can state the Linked Cluster Theorem

$$\boxed{\langle e^{\theta} \rangle = \exp\left\{ \langle e^{\theta} \rangle_c - 1 \right\}} \quad (38)$$

We can verify this to low order (say 3^{rd})

$$\left\langle 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots \right\rangle \stackrel{?}{=} \exp\left\{ \langle \theta \rangle_c + \frac{1}{2!}\langle \theta^2 \rangle_c + \frac{1}{3!}\langle \theta^3 \rangle_c + \cdots \right\}$$

$$1 + \langle \theta \rangle + \frac{\langle \theta^2 \rangle}{2!} + \frac{\langle \theta^3 \rangle}{3!} \stackrel{?}{=} 1 + \left[\langle \theta \rangle_c + \frac{1}{2!}\langle \theta^2 \rangle_c + \frac{1}{3!}\langle \theta^3 \rangle_c + \cdots \right]$$

$$+ \frac{1}{2!}\left[\langle \theta \rangle_c^2 + \langle \theta \rangle_c \langle \theta^2 \rangle_c + \cdots \right]$$

$$+ \frac{1}{3!}\left[\langle \theta \rangle_c^3 + \cdots \right]$$

$$\langle \theta \rangle + \frac{\langle \theta^2 \rangle}{2!} + \frac{\langle \theta^3 \rangle}{3!} + \cdots \stackrel{?}{=} \langle \theta \rangle_c + \frac{1}{2!}\left(\langle \theta^2 \rangle_c + \langle \theta \rangle_c^2 \right)$$

$$+ \frac{1}{3!}\left(\langle \theta^3 \rangle_c + 3\langle \theta \rangle_c \langle \theta^2 \rangle_c + \langle \theta \rangle_c^3 \right) + \cdots$$

Clearly $\langle \theta \rangle = \langle \theta \rangle_c$ because there are no subcorrelators. The 1^{st} order matches

$$\langle \theta^2 \rangle_c = \langle \theta^2 \rangle - \langle \theta \rangle^2 \Rightarrow \langle \theta^2 \rangle = \langle \theta^2 \rangle_c + \langle \theta \rangle_c^2$$

$\Rightarrow$ The second order matches

From (37)

$$\langle \theta^3 \rangle_c = \langle \theta^3 \rangle - 3\langle \theta \rangle \langle \theta^2 \rangle - \langle \theta \rangle^3$$

so the 3rd order also matches.

Now, from (31), (32), (33), (38)

$$Z = \int \mathcal{D}\phi(\bar{x})\, e^{-\frac{1}{T}(S_0 + V)}$$

$$= Z_0 \int \mathcal{D}\phi(\bar{x})\, e^{-\frac{S_0[\phi]}{T}} \frac{e^{-\frac{V[\phi]}{T}}}{Z_0}$$

(39)

$$\boxed{Z = Z_0 \left\langle e^{-\frac{V[\phi]}{T}} \right\rangle_0 = Z_0 \exp\left\{ \langle e^{-\frac{V}{T}} \rangle_{0c} - 1 \right\}}$$

where $\boxed{Z_0 = \int \mathcal{D}\phi(\bar{x})\, e^{-\frac{S_0[\phi]}{T}}}$ (40)

Z_0 is easy to compute by going to $\bar{k}$-space

$$\boxed{S_0 = \int \frac{d^dk}{(2\pi)^d} |\phi(\bar{k})|^2 \frac{(k^2 + r)}{2}} \quad (41)$$

Each $\bar{k}$ is idependent and one can consider $\boxed{\phi(\bar{k}) = \phi_1(\bar{k}) + i\, \phi_2(\bar{k})}$ (42)

ϕ_1, ϕ_2 real

To avoid overcounting we integrate .

$$\boxed{\int \mathcal{D}\phi(\bar{x}) = \prod_{\vec{k},\, k_x \geq 0}^{|\vec{k}|<\Lambda} \int_{-\infty}^{\infty} d\phi_1(\bar{k})\, d\phi_2(\bar{k})} \quad (43)$$

$$\boxed{\begin{aligned}\Lambda &= \text{cutoff} \\ &\simeq \frac{2\pi}{a}\end{aligned}} \quad (44)$$

$$\boxed{\int_{-\infty}^{\infty} d\phi_1\, e^{-\frac{\phi_1^2 (k^2+r)}{2T}} = \sqrt{\frac{2\pi T}{k^2+r}}} \quad (45)$$

An identical factor comes from the ϕ_2 integral

$$\Rightarrow \boxed{Z_0 = \prod_{\vec{k},\, k_x\geq 0}^{|\vec{k}|<\Lambda} \left[\frac{2\pi T}{\vec{k}^2+r}\right]} \quad (46)$$

$$= \text{Const}\; e^{-\sum_{\vec{k},\, k_x \geq 0} \ln\left(\frac{\vec{k}^2+r}{T}\right)}$$

$$\boxed{Z_0 = \text{Const}\; e^{-\sum_{k} \frac{1}{2} \ln\left(\frac{\vec{k}^2+r}{T}\right)} = e^{-\frac{F_0}{T}}} \quad (47)$$

$$\boxed{F_0 = (\text{Volume})\, \frac{T}{2} \int^{\Lambda} \frac{d^d k}{(2\pi)^d}\, \ln\left[\frac{\vec{k}^2+r}{T}\right]} \quad (48)$$

Where I have used $\sum_{\vec{k}} = \text{Volume} \int \frac{d^d k}{(2\pi)^d}$

We will need the mean-field result. Recall that for $r>0$ $\phi_0 = 0$, but

$$r<0 \qquad \phi_0 = \sqrt{\frac{-r}{\lambda}} = \sqrt{\frac{|r|}{\lambda}}$$

$$\Rightarrow \frac{F_{MF}}{T} = (\text{Volume}) \left[\frac{r\phi_0^2}{2} + \frac{\lambda \phi_0^4}{4}\right] = -\text{Volume}\, \frac{r^2}{4\lambda}$$

$$\boxed{\frac{F_{MF}}{T} = -\text{Volume}\;\; \Theta(-r)\, \frac{r^2}{4\lambda}} \quad (49)$$

Also, if $\phi(\bar{x}) = \phi_0 + \delta\phi(\bar{x})$, for $r<0$

$$-\frac{|r|}{2}(\phi_0+\delta\phi)^2 + \frac{\lambda}{4}(\phi_0+\delta\phi)^4$$

$$= -\frac{|r|}{2}\phi_0^2 - |r|\,\phi_0\delta\phi - \frac{|r|}{2}\delta\phi^2 + \frac{\lambda}{4}\left(\phi_0^4 + 4\phi_0^3\delta\phi + 6\phi_0^2\delta\phi^2 + \cdots\right)$$

$$= -\frac{r^2}{4\lambda} + (\delta\phi(\bar{x}))^2\left[-\frac{|r|}{2} + 6\phi_0^2\frac{\lambda}{4}\right] + \cdots$$

$$= \frac{-r^2}{4\lambda} + |r|(\delta\phi(\bar{x}))^2$$

So, for $r<0$

$$\boxed{\frac{F_0}{T} = -\text{Volume}\,\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\ln\left(k^2+2|r|\right)} \quad 50$$

So, the total free energy is

$$\Rightarrow \boxed{\begin{aligned}\frac{F}{T} &= \frac{F_{MF}}{T} + \frac{F_0}{T} + \cdots \\ &= \text{Volume}\left\{-\Theta(r)\frac{r^2}{4\lambda} + \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\ln\left[k^2+|r|\left(1+\Theta(-r)\right)\right] + \cdots\right\}\end{aligned}} \quad 51$$

Now consider the specific heat, which is

$$C_V \simeq \frac{1}{T}\frac{\partial^2 F}{\partial r^2} \qquad \text{for} \quad r<0$$

$$\frac{C_V}{\text{Vol}} \simeq \frac{\partial^2}{\partial r^2}\left\{-\frac{r^2}{4\lambda} + \frac{1}{2}\int^{\Lambda}\frac{d^dk}{(2\pi)^d}\ln\left(k^2+2|r|\right) + \cdots\right\}$$

$$\boxed{\frac{C_V}{Vol} = -\frac{1}{2\lambda} + 2\int^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+2|r|)^2}} \quad (52)$$

Now, for $d>4$ the integral is dominated by its upper limit, which means it is not sensitive to r. So, once again, the mean-field result stands.

However, if $d<4$ the integral is convergent in the UV (large k) so

$$d<4 \qquad \boxed{\int_0^{\Lambda} \frac{d^dk}{(k^2+2|r|)^2} \approx \int_0^{\infty} \frac{d^dk}{(k^2+2|r|)^2}} \quad (53)$$

Now we scale out $|r|$ $\quad k = \sqrt{2|r|}\, p$

$$= (2|r|)^{\frac{d}{2}-2} \int_0^{\infty} \frac{d^dp}{(p^2+1)^2}$$

$$\Rightarrow \boxed{\frac{C_V}{Vol} \simeq -\frac{1}{2\lambda} + (2|r|)^{\frac{d}{2}-2}} \quad (54)$$

Now, as $|r|\to 0$ the 2nd contribution diverges, which means the fluctuation contribution to C_V becomes larger than the mean-field contribution.

Fluctuations cannot be ignored for $d<4$

Thus, $d=4$ is the upper critical dimension

To get an approximate numerical criterion for the size of the critical regim where fluctuations dominate we set the size of the two contributions equal

$$\frac{1}{\lambda} = |r|^{\frac{d}{2}-2}$$

$$\Rightarrow \quad \boxed{|r_0| = \lambda^{\frac{2}{4-d}}} \quad (55)$$

This is the Ginzburg criterion. For $r > |r_0|$ we can trust mean-field theory, but for $r < |r_0|$ in $d < 4$ we cannot.