A review of Gaussian Integrals

Let us start with a 1D gaussian integral

$$\boxed{Z(a,J) = \int_{-\infty}^{\infty} dx \, e^{-\frac{a x^2}{2} - Jx}} \quad ①$$

complete squares in the exponent

$$-\frac{a x^2}{2} - Jx = -\frac{a}{2}\left[x^2 + \frac{2J}{a}x\right]$$

$$= -\frac{a}{2}\left[x^2 + \frac{2J}{a}x + \left(\frac{J}{a}\right)^2 - \left(\frac{J}{a}\right)^2\right]$$

$$\boxed{= -\frac{a}{2}\left[x + \frac{J}{a}\right]^2 + \frac{1}{2}\frac{J^2}{a}} \quad ②$$

Shift the variable $x \to x' = x + \frac{J}{a}$

$$Z(a,J) = e^{\frac{J^2}{2a}} \underbrace{\int_{-\infty}^{\infty} dx' \, e^{-\frac{a x'^2}{2}}}_{\sqrt{\frac{2\pi}{a}}}$$

$$\boxed{Z(a,J) = \sqrt{\frac{2\pi}{a}} \, e^{\frac{J^2}{2a}}} \quad ③$$

Now, the convenient thing is that the average of any power of x can be computed by taking derivatives of Z w.r.t. J and setting $J=0$

$$\boxed{\langle x^n \rangle = \int_{-\infty}^{\infty} dx\; x^n e^{-\frac{ax^2}{2}} = \frac{(-1)^n}{Z} \frac{\partial^n}{\partial J^n} Z(a,J) \Big|_{J=0}} \quad (4)$$

Now comes the idea of a connected correlator, which subtracts out any "sub-correlations". For example take $\langle x^4 \rangle$. There are sub-correlations $\langle x^2 \rangle^2$ hiding inside. How many $\langle x^2 \rangle^2$ are inside $\langle x^4 \rangle$? There are $\frac{1}{2}\binom{4}{2} = 3$ ways of selecting 2 pairs out of 4 x's. So we want

$$\boxed{\langle x^4 \rangle_c = \langle x^4 \rangle - 3 \langle x^2 \rangle \langle x^2 \rangle} \quad (5)$$

$$\langle x^2 \rangle = \left\{ e^{-J^2/2a} \sqrt{\frac{a}{2\pi}} \frac{\partial^2}{\partial J^2} \sqrt{\frac{2\pi}{a}}\, e^{J^2/2a} \right\} \Big|_{J=0}$$

$$= \left\{ e^{-J^2/2a} \frac{\partial^2}{\partial J^2} e^{J^2/2a} \right\} \Big|_{J=0}$$

$$= \left\{ e^{-J^2/2a} \frac{\partial}{\partial J} \left\{ \frac{J}{a} e^{J^2/2a} \right\} \right\} \Big|_{J=0}$$

$$= \left\{ e^{-J^2/2a} \left\{ \frac{1}{a} + \frac{J^2}{a^2} \right\} e^{J^2/2a} \right\} \Big|_{J=0} = \frac{1}{a}$$

$$\boxed{\langle x^2 \rangle = \frac{1}{a}} \quad (6)$$

How about $\langle x^4 \rangle$?

We need $\dfrac{\partial^4}{\partial J^4} e^{J^2/2a} = \dfrac{\partial^2}{\partial J^2}\left\{\left[\dfrac{1}{a} + \dfrac{J^2}{a^2}\right] e^{J^2/2a}\right\}$

$$= \frac{\partial}{\partial J}\left\{ \frac{J}{a}\left[\frac{1}{a} + \frac{J^2}{a^2}\right] e^{J^2/2a} + \frac{2J}{a^2} e^{J^2/2a}. \right\}$$

$$= \frac{\partial}{\partial J}\left\{\left[\frac{3J}{a^2} + \frac{J^3}{a^3}\right] e^{J^2/2a}\right\}$$

$$= \left\{\frac{3}{a^2} + \frac{3J^2}{a^3} + \frac{J}{a}\left(\frac{3J}{a} + \frac{J^3}{a^3}\right)\right\} e^{J^2/2a}$$

$$\Rightarrow \quad \boxed{\langle x^4 \rangle = \frac{3}{a^2}} \quad (7)$$

$$\Rightarrow \quad \boxed{\langle x^4 \rangle_c = \langle x^4 \rangle - 3\langle x^2\rangle\langle x^2\rangle = 0\ !!} \quad (8)$$

The compact way to write this is

$$\boxed{\langle x^4 \rangle_c = \frac{\partial^4}{\partial J^4} \ln Z[a,J]} \quad (9)$$

Convince yourself of this

In fact

$$\boxed{\langle x^n \rangle_c = \frac{\partial^n}{\partial J^n} \ln Z[a,J]} \quad (10)$$

Since $\ln Z[a,J]$ has only a quadratic dependence $J^2/2a$ we know that $\boxed{\langle x^n \rangle = 0 \quad n > 2}$ (11) This is Wick's Theorem in this simple example

Now let us consider a multivariable integral. Let x_i be the variables $i=1,\cdots,N$. Let the column vector of x_i be $\tilde{x}$ and that of J_i be $\tilde{J}$

$$\boxed{Z(\mathbb{A},\tilde{J}) = \int_{-\infty}^{\infty} dx_1 \cdots dx_N \, e^{-\frac{1}{2}\tilde{x}^T \mathbb{A}\tilde{x} - \tilde{J}^T\tilde{x}}} \quad (12)$$

Clearly only the symmetric part of $\mathbb{A}$ contributes, so we assume $\boxed{\mathbb{A}_{ij} = \mathbb{A}_{ji}}$ (13)

Since it is symmetric and real, we diagonalize it and obtain real eigenvalues a_α $\alpha=1,\cdots,N$. The unitary transformation

$\mathbb{U}_{i\alpha}$ satisfies

$$\boxed{\mathbb{A}_{ij}\,\mathbb{U}_{j\alpha} = a_\alpha \mathbb{U}_{i\alpha}} \quad (14)$$

$$\Rightarrow \boxed{(\mathbb{U}^+\mathbb{A}\,\mathbb{U})_{\alpha\beta} = a_\alpha \delta_{\alpha\beta}} \quad (15)$$

Make a change of variables

$$\boxed{\tilde{y} = \mathbb{U}\tilde{x}} \quad (16)$$

The Jacobian of the transformation is the (17) magnitude of $\det\mathbb{U}$. By unitarity $\boxed{|\det\mathbb{U}|=1}$

$$\Rightarrow \boxed{Z[\mathbb{A}, \tilde{J}] = \int_{-\infty}^{\infty} \left\{ \prod_\alpha dy_\alpha \right\} e^{-\frac{a_\alpha}{2} y_\alpha^2 - \tilde{J}^T \mathbb{U} \tilde{y}}} \quad (18)$$

Complete squares on each y_α, shift as before, and do the integrals to obtain

$$\boxed{Z[\mathbb{A}, \tilde{J}] = \frac{(2\pi)^{N/2}}{\sqrt{\prod_\alpha a_\alpha}} e^{\frac{1}{2} \tilde{J}^T \mathbb{A}^{-1} \tilde{J}} = \frac{(2\pi)^{N/2} e^{\frac{1}{2} \tilde{J}^T \mathbb{A} \tilde{J}}}{\sqrt{\det \mathbb{A}}}} \quad (19)$$

Now consider the correlators. From (12) it is easy to see

$$\langle x_i x_j \rangle = \frac{1}{Z[\mathbb{A}, 0]} \int_{-\infty}^{\infty} dx_1 \cdots dx_N \; x_i x_j \, e^{-\frac{1}{2} \tilde{x}^T \mathbb{A} \tilde{x}}$$

$$= \left\{ \frac{1}{Z[\mathbb{A}, \tilde{J}]} \frac{\partial^2}{\partial J_i \partial J_j} Z[\mathbb{A}, \tilde{J}] \right\} \Bigg|_{\tilde{J}=0} \quad (20)$$

due to the symmetry $\tilde{x} \to -\tilde{x}$ of the action

$$\boxed{\langle x_i \rangle = 0} \quad (21)$$

So

$$\langle x_i x_j \rangle = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle \quad (22)$$

$$= \langle x_i x_j \rangle_c = \left\{ \frac{\partial^2}{\partial J_i \partial J_j} \ln Z[\mathbb{A}, \tilde{J}] \right\} \Bigg|_{\tilde{J}=0}$$

Since
$$\boxed{\ln Z[\mathbb{A}, \tilde{J}] = \frac{N}{2}\ln(2\pi) - \frac{1}{2}\ln\{\det \mathbb{A}\} + \frac{1}{2}\tilde{J}^T \mathbb{A}^{-1}\tilde{J}} \quad (23)$$

It is clear that
$$\boxed{\langle x_i x_j \rangle = \langle x_i x_j \rangle_c = \mathbb{A}^{-1}_{ij}} \quad (24)$$

Secondly, since the dependence of $\ln Z$ is purely quadratic in J, we again get

Wick's Theorem

$$\boxed{\begin{aligned} &\langle x_i x_j x_k x_l \rangle_c = 0 \quad \Rightarrow \\ &\langle x_i x_j x_k x_l \rangle = \langle x_i x_j \rangle \langle x_k x_l \rangle + \langle x_i x_k \rangle \langle x_j x_l \rangle + \langle x_i x_l \rangle \langle x_j x_k \rangle \\ &\qquad = \mathbb{A}^{-1}_{ij}\mathbb{A}^{-1}_{kl} + \mathbb{A}^{-1}_{ik}\mathbb{A}^{-1}_{jl} + \mathbb{A}^{-1}_{il}\mathbb{A}^{-1}_{jk} \end{aligned}} \quad (25)$$

and similar statements for all higher correlators.

Now let us go from a discrete set of variables x_i to a continuous set of fields $\phi(\vec{x})$

$$\boxed{x_i \longrightarrow \phi(\vec{x})} \quad (26)$$

The i and position $\vec{x}$ are labels, but

x and ϕ are variables being integrated

$$\boxed{\int dx_1 \cdots dx_N \Rightarrow \int \mathcal{D}\phi(\vec{x}) = \int_{-\infty}^{\infty} \prod_{\vec{x}} d\phi(\vec{x})} \quad (27)$$

To make sense of a "product" over a continuous set of points $\vec{x}$, one needs to impose a lattice so that it becomes a discrete product, and let the lattice spacing vanish in the limit.

How about $\mathbb{A}_{ij}$? This goes to an integral kernel $K(\vec{x},\vec{x}')$

$$\boxed{\mathbb{A}_{ij} \longrightarrow K(\vec{x},\vec{x}')} \quad (28)$$

$$\boxed{S = \frac{1}{2} x_i \mathbb{A}_{ij} x_j \longrightarrow \int d^d\vec{x}\, d^d\vec{x}'\, \frac{1}{2}\phi(\vec{x}) K(\vec{x},\vec{x}')\phi(\vec{x}')} \quad (29)$$

The Kernel we normally have is

$$\boxed{K(\vec{x},\vec{x}') = K(\vec{x}-\vec{x}') = \left[-\vec{\nabla}_{\vec{x}}^2 + r\right] \delta^d(\vec{x}-\vec{x}')} \quad (30)$$

$$\boxed{\begin{aligned} S &= \int d^dx \quad \frac{1}{2}\phi(\vec{x})\left[-\vec{\nabla}^2 + r\right]\phi(\vec{x}) \\ &= \int d^dx \quad \frac{1}{2}\left[(\vec{\nabla}\phi)^2 + r\phi^2\right] \end{aligned}} \quad (31)$$

provided ϕ vanishes sufficiently rapidly at ∞

In the correlators we encountered A^{-1}. The analogue of A^{-1} is the Green's function $G(\bar{x},\bar{x}')$ of $K(\bar{x},\bar{x}')$

$$\boxed{\int d^d\bar{x}''\, G(\bar{x},\bar{x}'')K(\bar{x}'',\bar{x}') = \int d^d\bar{x}''\, K(\bar{x},\bar{x}'')G(\bar{x}'',\bar{x}') = \delta^d(\bar{x}-\bar{x}')} \quad (32)$$

Plugging in our favorite Kernel (30)

we have $\boxed{(-\bar{\nabla}^2 + r)\, G(\bar{x},\bar{x}') = \delta^d(\bar{x}-\bar{x}')}$ (33)

By translation invariance $G = G(\bar{x}-\bar{x}')$

Take a Fourier transform

$$\boxed{G(\bar{x}-\bar{x}') = \int \frac{d^d k}{(2\pi)^d}\, G(\bar{k})\, e^{i\bar{k}\cdot(\bar{x}-\bar{x}')}} \quad (34)$$

$$\Rightarrow \boxed{(k^2+r)\, G(\bar{k}) = 1} \quad (35)$$

$$\boxed{G(\bar{k}) = \frac{1}{k^2+r}} \quad (36)$$

$$\boxed{G(\bar{x}-\bar{x}') = \int \frac{d^d k}{(2\pi)^d}\, \frac{e^{i\bar{k}\cdot(\bar{x}-\bar{x}')}}{k^2+r}} \quad (37)$$

By the logic leading up to (24) we get

$$\boxed{\langle \phi(\bar{x})\, \phi(0) \rangle = G(\bar{x})} \quad (38)$$