Wilsonian RG

By now we know the following facts

1) At a continuous phase transition there is scale invariance

2) Integrating fast degrees of freedom leads to a flow in Hamiltonian space. The most natural way to achieve scale invariance is for the flow to have a fixed point

3) Near the fixed point, the linearized flow map has eigendirections with eigenvalues greater than 1 (relevant) and less than 1 (irrelevant). The number of relevant eigenvalues is the number of parameters to be tuned to reach criticality.

4) The relevant eigenvalues control the critical exponents via the scaling relations

5) For theories with $\lambda\phi^4$ coupling, one cannot do perturbation theory in λ below the upper critical dimension of 4.

What we need is a way to calculate the exponents in some controlled approximation. This is what Wilson's momentum shell RG achieves.

As usual, we start with the Ising Model coarse-grained action

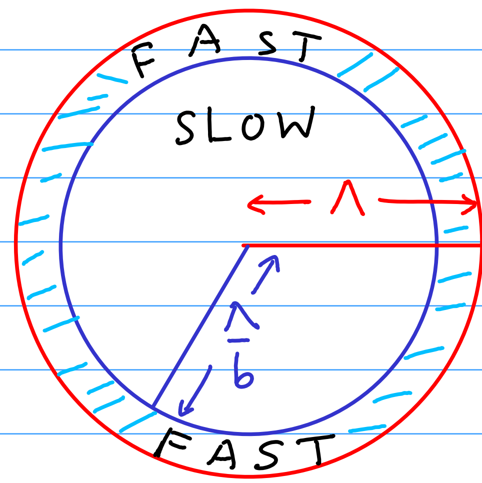
$$\boxed{S = \int d^d x \left\{ \frac{1}{2}(\bar{\nabla}\phi)^2 + \frac{r}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \right\}} \quad (1)$$

with

$$\boxed{\phi(\bar{x}) = \int_0^{\Lambda} \frac{d^d k}{(2\pi)^d}\, \tilde{\phi}(\bar{k})\, e^{i\bar{k}\cdot\bar{x}}} \quad (2)$$

Our "Brillouin Zone" is a sphere of radius Λ in $\bar{k}$-space

Divide the BZ into a shell of "fast" modes with



$$\boxed{\frac{\Lambda}{b} \leq k \leq \Lambda} \quad (3)$$

and a sphere of slow modes

$$\boxed{k \leq \frac{\Lambda}{b}} \quad (4)$$

slow fast

$$\boxed{\phi(\bar{x}) = \phi_<(\bar{x}) + \phi_>(\bar{x})} \quad (5)$$

$$\boxed{Z = \int \mathcal{D}\phi\, e^{-\frac{S}{T}} = \int \mathcal{D}\phi_<(\bar{x}) \int \mathcal{D}\phi_>(\bar{x})\, e^{-\frac{S}{T}}} \quad (6)$$

Using (5) break up (1) into slow-only, fast-only, and mixed pieces

$$S = \int d^d x \left\{ \frac{1}{2}\left[(\vec{\nabla}\phi_<)^2 + r\phi_<^2\right] + \frac{\lambda}{4}\phi_<^4 \right.$$ $S_<[\phi_<]$

$$+ \frac{1}{2}\left[(\vec{\nabla}\phi_>)^2 + r\phi_>^2\right] + \frac{\lambda}{4}\phi_>^4$$ $S_>[\phi_>]$

$$\left. + \frac{\lambda}{4}\left[4\phi_<\phi_>^3 + 6\phi_<^2\phi_>^2 + 4\phi_<^3\phi_>\right] \right\}$$ $S_{int}[\phi_<,\phi_>]$ (7)

Now we need to do

$$\int \mathcal{D}\phi_>(\bar{x})\, e^{-\frac{S_>[\phi_>] + S_{int}[\phi_<,\phi_>]}{T}} \qquad (8)$$

Separate out the quadratic part of $S_>$ and rewrite as

$$S_>[\phi_>] + S_{int}[\phi_<,\phi_>] = \underbrace{\frac{1}{2}\int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(2\pi)^d} |\tilde{\phi}_>(k)|^2 (k^2 + r)}_{S_{0>}} + V[\phi_<,\phi_>] \qquad (9)$$

$$V[\phi_<,\phi_>] = S_{int}[\phi_<,\phi_>] + \frac{\lambda}{4}\int d^d x\, \phi^4(\bar{x}) \qquad (10)$$

So (8) can be rewritten as

$$\int \mathcal{D}\phi_>(\bar{x})\, e^{-\frac{S_{0>}}{T}} e^{-\frac{V[\phi_<,\phi_>]}{T}} = Z_{0>} \left\langle e^{-\frac{V[\phi_<,\phi_>]}{T}} \right\rangle_{0>} \qquad (11)$$

$$Z_{0>} = \exp\left\{ -\text{Volume} \int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(2\pi)^d} \ln\left(\frac{\vec{k}^2 + r}{T}\right) \right\} \qquad (12)$$

Now we use the linked cluster theorem to obtain

$$\boxed{\int \mathcal{D}\phi_> \, e^{-\frac{S_> + S_{int}}{T}} = \exp\left\{ -L^d \int \frac{d^dk}{(2\pi)^d} \ln\left[\frac{k^2+r}{T}\right] + \left\langle e^{-\frac{V[\phi_<,\phi_>]}{T}} \right\rangle_{0>c} - 1 \right\}} \quad (13)$$

$\langle \; \rangle_{0>c}$ indicates an average over configurations of $\phi_>$, in the quadratic action $S_{0>}$, and the further subscript c indicates that cumulants must be considered

$$\boxed{= \exp\left\{ -L^d \int \frac{d^dk}{(2\pi)^d} \ln\left[\frac{k^2+r}{T}\right] - \frac{1}{T}\langle V \rangle_{0>c} + \frac{1}{2T^2}\langle V^2 \rangle_{0>c} + \cdots \right\}} \quad (14)$$

Let us consider 1st something that appears trivial, namely, zeroth order in λ in this $\phi_>$ integral.

So the action $S_<$ is not changed to this order.

$$\boxed{S_< = \int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d} |\phi_<(\bar{k})|^2 \left(\frac{k^2+r}{2}\right) + \frac{\lambda}{4} \int d^dx \, \phi_<^4(\bar{x})} \quad (15)$$

However, now Wilson did something very clever in order to make the new problem as similar as possible to the old.

We rescaled momentum so that the upper limit is again Λ.

Define $\boxed{\vec{p} = b\vec{k}}$ (16) $\boxed{\vec{k} = \frac{\vec{p}}{b}}$ (17)

Look 1st at the k^2 part only

(18)
$$\boxed{\int_0^{\Lambda/b} \frac{d^d k}{(2\pi)^d} \, k^2 \left|\tilde{\phi}(\bar{k})\right|^2 = b^{-(d+2)} \int_0^{\Lambda} \frac{d^d p}{(2\pi)^d} \, \vec{p}^{\,2} \left|\tilde{\phi}(\vec{p}/b)\right|^2}$$

We want to make this look like the original k^2 term in the full BZ.

Define $\boxed{\tilde{\phi}'(\vec{p}) = b^{-\frac{(d+2)}{2}} \tilde{\phi}(\vec{p}/b)}$ (19) Field rescaling

Then
$$\boxed{\int_0^{\Lambda/b} \frac{d^d k}{(2\pi)^d} \, k^2 \left|\tilde{\phi}(\bar{k})\right|^2 = \int_0^{\Lambda} \frac{d^d p}{(2\pi)^d} \, \bar{p}^{\,2} \left|\tilde{\phi}'(\bar{p})\right|^2} \quad (20)$$

How does $\phi(\bar{x})$ rescale under this transformation?

$$\phi_<(\bar{x}) = \int_0^{\Lambda/b} \frac{d^d k}{(2\pi)^d} \, \tilde{\phi}(\bar{k}) \, e^{i\bar{k}\cdot\bar{x}} = b^{-d} \int_0^{\Lambda} \frac{d^d p}{(2\pi)^d} \, \tilde{\phi}(\vec{p}/b) \, e^{i\bar{p}\cdot\vec{x}/b}$$

Now $\tilde{\phi}(\bar{p}/b) = \tilde{\phi}'(\bar{p}) \, b^{\frac{d+2}{2}}$

(21)
$$\boxed{\phi_<(\bar{x}) = b^{1-\frac{d}{2}} \int_0^{\Lambda} \frac{d^d p}{(2\pi)^d} \, \tilde{\phi}'(\bar{p}) e^{i\bar{p}\cdot\vec{x}/b} = b^{1-\frac{d}{2}} \phi'(\bar{x}/b)}$$

So in real space the transformation looks like

$$\boxed{\vec{x} \to \vec{x}' = \vec{x}/b} \quad (22) \qquad \boxed{\int d^d x = \int d^d x'\, b^d} \quad (23)$$

$$\boxed{\begin{aligned} \phi(\bar{x}) &= b^{1-\frac{d}{2}}\, \phi'(\bar{x}') \\ \bar{\nabla}\phi(\bar{x}) &= b^{-d/2}\, \nabla' \phi'(\bar{x}') \end{aligned}} \quad (24)$$

$$\Rightarrow \boxed{\begin{aligned} \int d^d x\, (\bar{\nabla}\phi)^2 &= \int d^d \bar{x}'\, b^d \left[b^{-d/2}\, \bar{\nabla}' \phi'(\bar{x}') \right]^2 \\ &= \int d^d x' \left(\bar{\nabla}' \phi'(\bar{x}') \right)^2 \end{aligned}} \quad (25)$$

Now let us see how the other terms behave

$$\boxed{\begin{aligned} \int d^d x\, \phi^2 r &= \int d^d x'\, b^d \left(\phi'(\bar{x}')\right)^2 b^{2-d}\, r \\ &= \int d^d x'\, \phi'^2(\bar{x}')\, (r b^2) \end{aligned}} \quad (26)$$

So under this transformation

$$\boxed{r' = r b^2} \quad (27)$$

How about $\lambda \int d^d x\, \phi^4(\bar{x})$?

$$\boxed{\begin{aligned} \lambda \int d^d x\, \phi^4(\bar{x}) &= \lambda \int d^d \bar{x}'\, b^d \left[b^{1-\frac{d}{2}}\, \phi'(\bar{x}') \right]^4 \\ &= \lambda\, b^{4-d} \int d^d \bar{x}' \left(\phi'(\bar{x}')\right)^4 \end{aligned}} \quad (28)$$

Thus $\boxed{\lambda' = \lambda\, b^{4-d}}$ (29)

Since we are working to zeroth order in perturbations in λ, this is known as tree level (no loops yet). (27), (29) are the tree-level RG eq^{n}s

$$\boxed{\begin{aligned} r' &= r b^2 \\ \lambda' &= \lambda\, b^{4-d} \end{aligned}} \quad (30)$$

The λ eq^n shows that if $d > 4$ λ shrinks and is irrelevant (at tree level). This is why for $d > 4$ we can carry out perturbation theory.

For $d < 4$ λ grows, so one cannot do perturbation theory.

This is a good time to address another point. We know that when we coarse-grain we get not only the terms we have kept, but also others, such as

$$\boxed{g_6 \phi^6 + g_8 \phi^8 + \cdots} \quad (31)$$

and higher derivative terms

$$\boxed{(\bar{\nabla}^2 \phi)^2, \quad \left[(\bar{\nabla}\phi)^2\right]^2, \text{ etc}} \quad (32)$$

Why did we not keep them in S? Because they are irrelevant even at tree-level!

Consider $\int g_{2n}\, d^dx\, \phi^{2n}(\bar{x})$ and ask how it transforms under the tree-level RG

$$\boxed{\begin{aligned}\int g_{2n}\, d^dx\, \phi^{2n}(\bar{x}) &= \int g_{2n}\, d^dx'\, b^d \left[b^{1-\frac{d}{2}}\phi'(\bar{x}')\right]^{2n} \\ &= g_{2n}\, b^{2n-(n-1)d} \int d^dx' \left(\phi'(\bar{x}')\right)^{2n}\end{aligned}} \quad (33)$$

$$\Rightarrow \boxed{g_{2n}' = g_{2n}\, b^{2n-(n-1)d}} \quad (34)$$

Near $d=4$ this becomes b^{-2n+4} which shrinks for $n>2 \Rightarrow \phi^6, \phi^8$ are irrelevant.

Similarly

$$\boxed{\begin{aligned}\int d^dx\, (\bar{\nabla}^2\phi)^2 &= \int d^dx'\, b^d \left[b^{-1-\frac{d}{2}}\bar{\nabla}'^2\phi'(\bar{x}')\right]^2 \\ &= b^{-2}\int d^dx'\, \nabla'^2\phi'(\bar{x}')\end{aligned}} \quad (35)$$

irrelevant!

This brings us to the notion of the naive scaling dimension (also often called the engineering dimension) of a term in the action. This refers to the power of b that it acquires under the tree-level RG

Each power of $\bar{x}$ $\rightarrow$ Length dimension $+1$
Scaling dimension -1

Each gradient $\rightarrow$ Length dimension -1
Scaling dimension $+1$

Each field $\phi(x)$ $\rightarrow$ Length dimension $1-\frac{d}{2}$
Scaling dimension $\frac{d}{2}-1$

Multiplying fields adds scaling dimensions

So $\int d^dx \ \phi^4(x)$ has scaling dimension $d-4\left(\frac{d}{2}-1\right)$

$= 4-d$

If the scaling dimension is positive, that term grows under tree-level RG, while if it is negative it shrinks, and is irrelevant.

So, tree-level RG tells us

$$r(b) = b^2 r$$
$$\lambda(b) = b^{4-d}\lambda$$
$$g_{2n}(b) = b^{d-n(d-2)} g_{2n}$$

Is there a fixed point? Yes!

$$\boxed{r^* = \lambda^* = g_{2n}^* = 0} \quad (36)$$

This is known as the Gaussian fixed point
For $d>4$ this has r and h as relevant directions, while for $d<4$ λ is also relevant.

For $\lambda>0$ the Gaussian fixed pt is unstable in $d<4$

So far we have ignored the effect of integrating fast modes. The coupling between $\phi_>$, $\phi_<$ arises due to λ.

To begin, let us go to 1st order in λ. From (14) we need to do

$$\boxed{\langle \bar{V} \rangle_{0>c} = \langle \bar{V} \rangle_{0>}} \quad \text{no proper partitions}$$

(37)

$$\boxed{= \frac{\lambda}{4} \int d^4x \left\{ \langle \phi_>^4(\bar{x}) \rangle + 4\, \phi_<(\bar{x}) \langle \phi_>^3(\bar{x}) \rangle + 6\, \phi_<^2 \langle \phi_>^2(\bar{x}) \rangle + 4\, \phi_<^3(\bar{x}) \langle \phi_>(\bar{x}) \rangle \right\}}$$

We are only interested in the parts that contain $\phi_<$, because those will renormalize $S_<$. Also, $\boxed{\langle \phi_> \rangle = \langle \phi_>^3 \rangle = 0}$ by symmetry (38)

So $$\boxed{\langle \bar{V} \rangle_{0>} = \frac{3}{2} \lambda \int d^dx \quad \phi_<^2(\bar{x}) \langle \phi_>^2(\bar{x}) \rangle_{0>}} \quad (39)$$

$$\boxed{\langle \phi_>^2(\bar{x}) \rangle_{0>} = T \int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + r}} \quad (40)$$

For $r \ll \Lambda/b$ we can expand r on the RHS, as we will do soon.

Now let us go to spherical coordinates in d dimensions

$$\boxed{d^d k = k^{d-1}\, dk\, d^{d-1}\Omega_k} \quad (41)$$

$\Omega_{\bar{k}}$ are the angular variables

The integrand does not depend on angles so we need to do

$$\boxed{\int d^{d-1}\Omega_k = S_{d-1}} \quad (42)$$

"Area" of $d-1$ sphere

Do this by the following trick. Consider

$$\boxed{\int d^d k\; e^{-\frac{a\vec{k}^2}{2}} = I(a,d)} \quad (43)$$

By doing all the components separately we get

$$\boxed{I(a,d) = \left(\sqrt{\frac{2\pi}{a}}\right)^d = \left(\frac{2\pi}{a}\right)^{\frac{d}{2}}} \quad (44)$$

On the other hand, first go to spherical coordinates to obtain

$$\boxed{I(a,d) = \int_0^\infty k^{d-1}\, dk\; e^{-\frac{a}{2}k^2} \int d^{d-1}\Omega_k} \quad (45)$$

$$\boxed{\frac{a}{2}k^2 = t} \quad (46) \qquad a k\, dk = dt \qquad k = \left(\frac{2t}{a}\right)^{\frac{1}{2}}$$

$$k^{d-1}\, dk = \frac{k^{d-2}}{a}\, a k\, dk = \frac{1}{a}\left(\frac{2t}{a}\right)^{\frac{d-2}{2}} dt$$

$$I(a,d) = \frac{1}{a^{d/2}} \, 2^{\frac{d-2}{2}} \underbrace{\int_0^\infty dt \; t^{\frac{d}{2}-1} e^{-t}}_{\Gamma(\frac{d}{2})} \int d^{d-1}\Omega_k$$

$$\boxed{I(a,d) = \frac{2^{\frac{d-2}{2}}}{a^{d/2}} \Gamma(d/2) \int d^{d-1}\Omega_k} \quad (47)$$

Comparing, we get

$$\boxed{S_{d-1} = \int d^{d-1}\Omega_k = \frac{2\pi^{d/2}}{\Gamma(d/2)}} \quad (48)$$

Check $\quad d=3 \quad \Rightarrow \quad \dfrac{2\pi^{3/2}}{\Gamma(\frac{3}{2})}$

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi} \quad \Rightarrow \quad \int d^2\Omega = 4\pi$$

Back to (39), (40)

$$\boxed{\langle V[\phi_<, \phi_>]\rangle_{>0} = \frac{3}{2}\lambda T \int d^d x \; \phi_<^2(\bar{x}) \; S_{d-1} \int_{\Lambda/b}^{\Lambda} \frac{k^{d-1} dk}{k^2+r}} \quad (49)$$

For $\quad r \ll \dfrac{\Lambda}{b}$ we expand

$$\boxed{\int_{\Lambda/b}^{\Lambda} \frac{k^{d-1} dk}{k^2+r} \approx \int_{\Lambda/b}^{\Lambda} k^{d-3} dk \left[1 - \frac{r}{k^2} + \cdots\right]} \quad (50)$$

$$\boxed{\int_{\Lambda/b}^{\Lambda} \frac{k^{d-1}\,dk}{k^2+r} = \frac{\Lambda^{d-2}}{d-2}\left[1-\frac{1}{b^{d-2}}\right] - r\,\frac{\Lambda^{d-4}}{d-4}\left[1-\frac{1}{b^{d-4}}\right] + \cdots} \quad (51)$$

Since this term comes with a $\phi_<^2(\bar{x})$ it renormalizes the pre-existing $\frac{r}{2}\phi_<^2$.

Therefore, there is a 1^{st}-order in λ contribution

$$\boxed{\delta r = 3\lambda T\, S_{d-1}\left\{\frac{\Lambda^{d-2}}{d-2}\left[1-\frac{1}{b^{d-2}}\right] - r\,\frac{\Lambda^{d-4}}{d-4}\left[1-\frac{1}{b^{d-4}}\right]\right\}} \quad (52)$$

Thus, we must "improve" (27) to include this

$$\boxed{r' = r(b) = b^2\left\{r + 3\lambda T S_{d-1}\frac{\Lambda^{d-2}}{d-2}\left[1-b^{2-d}\right] - \frac{3\lambda r T\Lambda^{d-4}}{d-4}\left[1-b^{4-d}\right]\right\}} \quad (53)$$

All these powers of b and Λ are getting quite cumbersome, so let us make some simplifications. First recall that r has scaling dimension 2.

So does $\boxed{\Lambda^2 = \left(\frac{\Lambda}{b}\right)^2 b^2 = (\Lambda(b))^2 b^2}$ (54)

Let us define a dimensionless quantity

$\boxed{\tilde{r} = \frac{r}{\Lambda_0^2}}$ (55) where Λ_0 is the original cutoff

Similarly λ has scaling dimension $4-d$, so define

$$\boxed{\tilde{\lambda} = \frac{\lambda}{\Lambda_0^{4-d}}} \quad (56)$$

$$\Rightarrow \lambda \Lambda_0^{d-2} = \frac{\lambda}{\Lambda_0^{4-d}} \Lambda_0^2 = \tilde{\lambda} \Lambda_0^2$$

$$\lambda r \Lambda_0^{d-4} = \tilde{\lambda} \tilde{r} \Lambda_0^2 \quad (57)$$

$$\Rightarrow \boxed{\tilde{r}(b) = b^2 \left\{ \tilde{r} + \frac{3\tilde{\lambda} T S_{d-1}}{d-2} \left[1 - b^{2-d} \right] - \frac{3\tilde{\lambda}\tilde{r} T S_{d-1}}{d-4} \left[1 - b^{4-d} \right] \right\}}$$

The second simplification is to make b infinitesimally different from 1

$$\boxed{b = 1 + \delta b \qquad \delta b \to 0^+} \quad (58)$$

$$\boxed{\tilde{r}(b) = \tilde{r}(1+\delta b) = \tilde{r} + \delta b \frac{d\tilde{r}}{db} + \cdots} \quad (59)$$

$$\frac{1-b^{2-d}}{d-2} = \delta b = \frac{1-b^{4-d}}{d-4} \quad \text{to } 1^{st} \text{ order}$$

$$\Rightarrow \boxed{\frac{d\tilde{r}}{db} = 2\tilde{r} + 3\tilde{\lambda} T S_{d-1} - 3\tilde{\lambda}\tilde{r} T S_{d-1}} \quad (60)$$

This is an example of a Differential RG flow equation. The object on the RHS is called the beta function of $\tilde{r}$.

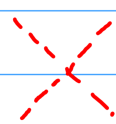
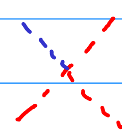
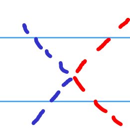
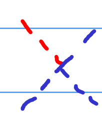
To obtain a similar eq^n for the flow of $\tilde{\lambda}$ we need to go to second order in λ in (14), and look at

(61)
$$\boxed{\frac{1}{2T^2}\left\langle \bar{V}[\phi_<,\phi_>]^2\right\rangle_{0>c} = \frac{1}{2T^2}\left\{\langle \bar{V}^2\rangle_{0>} - \langle \bar{V}\rangle_{0>}^2\right\}}$$

The subtraction of $\langle \bar{V}\rangle_{0>}^2$ from $\langle \bar{V}^2\rangle_{0>}$ means that only terms in which at least one $\phi_>$ from two different $\bar{V}$'s are correlated are to be kept.

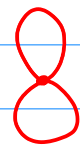
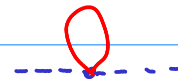
A convenient way to visualize this is Feynman diagrams. Each $\bar{V}$ contains 4 ϕ's. Let us denote $\phi_<$ by blue lines and $\phi_>$ by red lines.

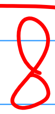
$\bar{V}$ has the following vertices (62)

$\bar{V} = \frac{\lambda}{4}\{$  $+ 4$  $+ 6$  $+ 4$  $\}$

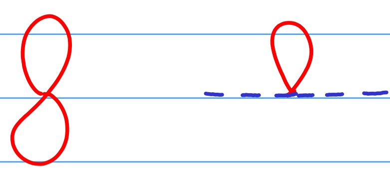
A correlator $\langle \phi_>(\bar{x})\,\phi_>(\bar{x}')\rangle$ can be denoted by a solid red line.

Integrating the $\phi_>(\bar{x})$ fields means that all $\phi_>$'s must be part of correlators, and none of them can be left dangling from the vertices. On the other hand all $\phi_<$'s must be left dangling

$$\boxed{\langle V\rangle_{0>} = \frac{\lambda}{4}\Big\{3 \quad}$$  $+\,6$  $\Big\}$ (63)

where  arises from the 3 ways of "contracting" (or joining up) the 4 $\phi_>$'s in the 1st term of V. The second term is precisely what we computed in (64)

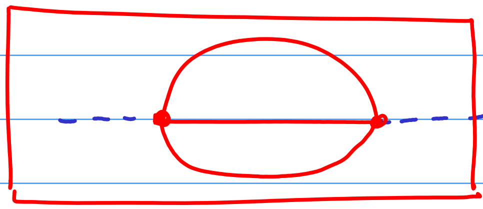
Now go to $\langle V^2\rangle_{0>}$. Now there are two vertices involved. If the contractions do not connect the two vertices, such as



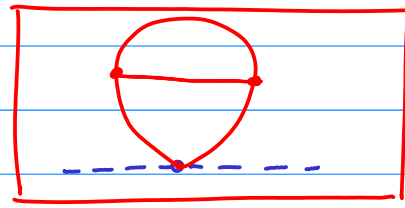
this corresponds to $\langle V\rangle_{0>}^2$ and is subtracted out. So we keep only terms where the two vertices are connected by at least one red line. There are several types of terms, which we will order by the number of $\phi_<$ (dangling blue dotted lines) they have.

Terms which have no $\phi_<$ contribute to the regular part of F, and can be ignored.

Terms which contain 2 $\phi_<$'s are

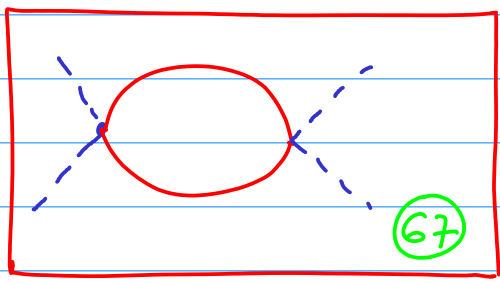


(65)

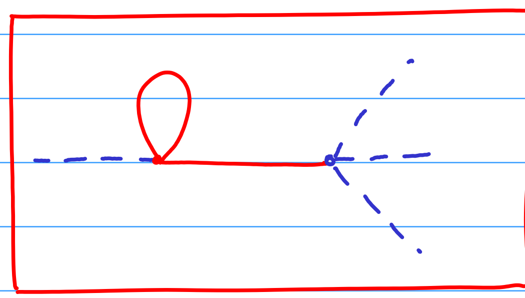


(66)

Terms containing 4 $\phi_<$'s are

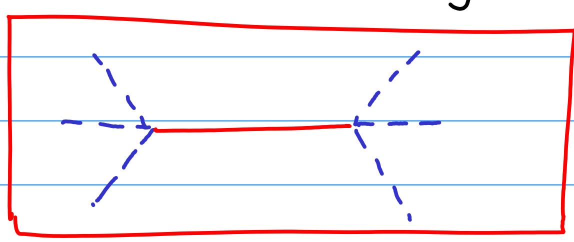


(67)



(68)

Terms containing 6 $\phi_<$'s are



(69)

To proceed further we make the momenta of $\phi_<$ "external legs" tend to zero.

The rationale is that originally λ is momentum independent, but under RG it may acquire $\bar{k}$-dependence. However, by arguments identical to those leading to (34)-(35) one can show that the $\bar{k}$-dependence is irrelevant at tree-level.

Setting the $\bar{k}$ of the external legs to 0, it is clear by momentum conservation at each vertex that diagrams 68, 69 vanish

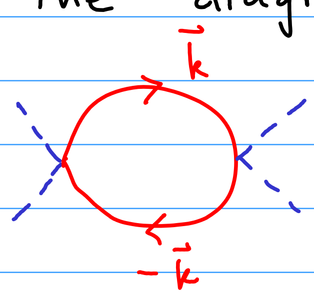
Diagram 66 again leads to a shift of T_c and can be ignored. Diagram 65 is a bit more subtle, and causes something called wave-f^n renormalization, or field renormalization. The Wilsonian RG is not the best way to handle this, and we will treat this in the Field-theoretic RG later.

This leaves diagram 67 which is the lowest nontrivial correction to the quartic coupling λ

First the numerical factors that multiply this diagram.

$$\boxed{\frac{1}{2T^2}\left(\frac{3\lambda}{2}\right)^2 2} \tag{70}$$

Now the diagram itself



$$= \boxed{\int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d}\,\frac{T^2}{(k^2+r)^2}} \tag{71}$$

For small $r \ll \frac{\Lambda}{b}$ ignore r in the denominator

$$\boxed{S_{d-1} \int_{\Lambda/b}^{\Lambda} k^{d-5}\, dk = S_{d-1} \frac{\Lambda^{d-4}}{d-4}\left[1 - \frac{1}{b^{d-4}}\right]} \quad (72)$$

From (14) this term appears with a + sign in the exponent, so its contribution to $S_<$ will come with a negative sign

$$\Rightarrow \boxed{\delta\left(\frac{\lambda}{4}\right) = -T\left(\frac{3\lambda}{2}\right)^2 S_{d-1} \frac{\Lambda^{d-4}}{d-4}\left[1 - b^{4-d}\right]} \quad (73)$$

So, Eq (29) should be corrected to

$$\boxed{\lambda' = b^{4-d}\left[\lambda - 9\lambda^2 T S_{d-1} \frac{\Lambda^{d-4}}{d-4}\left[1 - b^{4-d}\right]\right]} \quad (74)$$

Make λ dimensionless $\tilde{\lambda} = \frac{\lambda}{\Lambda^{4-d}}$ and

cast the flow into differential form by letting $b = 1 + \delta b$ $\delta b \to 0^+$

$$\boxed{\frac{d\tilde{\lambda}}{db} = (4-d)\tilde{\lambda} - 9\tilde{\lambda}^2 T S_{d-1}} \quad (75)$$

The right hand side is called the beta function of the coupling λ, $\beta(\lambda)$

At the moment we only have $\beta(\lambda)$ to order λ^2. However, there is a very important property of $\beta(\lambda)$: It is analytic in λ. This is because in obtaining it we have not integrated out slow fluctuations. So we can hope that for small λ this β-function is correct. A way to formalize and use this is to consider d to be a continuous variable, and work "near" 4 dimensions

Assume $\boxed{\epsilon = 4-d \ll 1}$ (The ϵ-expansion) (76)

The flows we care about are

$$\boxed{\frac{d\tilde{r}}{db} = 2\tilde{r} + 3\tilde{\lambda} T S_{d-1} - 3\tilde{\lambda}\tilde{r} T S_{d-1}} \quad (60)$$

$$\boxed{\frac{d\tilde{\lambda}}{db} = \epsilon\tilde{\lambda} - 9\tilde{\lambda}^2 T S_{d-1}} \quad (77)$$

Of course the Gaussian fixed point $r_G^* = \lambda_G^* = 0$ remains a fixed point,

but now there is a nontrivial Wilson-Fisher fixed point

$$\boxed{\tilde{\lambda}^* = \frac{\epsilon}{9 T S_{d-1}} \qquad \tilde{r}^* = -\frac{\epsilon}{6[1-\epsilon/6]}} \quad (78)$$

Now we need to linearize the flow around the fixed pt and find the eigenvalues of the linearized flow matrix

$$\boxed{\tilde{r} = \tilde{r}^* + \delta\tilde{r} \qquad \tilde{\lambda} = \tilde{\lambda}^* + \delta\tilde{\lambda}} \quad (79)$$

$$\boxed{\frac{d\tilde{r}}{db} = \beta_r(\tilde{r},\tilde{\lambda}) \qquad \frac{d\tilde{\lambda}}{db} = \beta_\lambda(\tilde{r},\tilde{\lambda})} \quad (80)$$

$$\boxed{\beta_r(\tilde{r}^*,\tilde{\lambda}^*) = \beta_\lambda(\tilde{r}^*,\tilde{\lambda}^*) = 0} \quad (81)$$

$$\boxed{\frac{d(\tilde{r}^*+\delta\tilde{r})}{db} = \frac{d\delta\tilde{r}}{db} = \beta_r(\tilde{r}^*+\delta\tilde{r},\tilde{\lambda}^*+\delta\tilde{\lambda}) = \delta\tilde{r}\,\frac{\partial\beta_r}{\partial\tilde{r}}\bigg|_* + \delta\tilde{\lambda}\,\frac{\partial\beta_r}{\partial\tilde{\lambda}}\bigg|_*} \quad (82)$$

and
$$\boxed{\frac{d\delta\tilde{\lambda}}{db} = \delta\tilde{r}\,\frac{\partial\beta_\lambda}{\partial\tilde{r}}\bigg|_* + \delta\tilde{\lambda}\,\frac{\partial\beta_\lambda}{\partial\tilde{\lambda}}\bigg|_*} \quad (83)$$

$$\Rightarrow \boxed{\begin{aligned} \frac{d\,\delta\tilde{r}}{db} &= \delta\tilde{r}\left[2 - 3\tilde{\lambda}^* T S_{d-1}\right] + \delta\tilde{\lambda}\, 3 T S_{d-1}(1-\tilde{r}^*) \\ \frac{d}{db}\,\delta\tilde{\lambda} &= \left(\epsilon - 18\tilde{\lambda}^* T S_{d-1}\right)\delta\tilde{\lambda} \end{aligned}} \quad (84)$$

So
$$\boxed{\frac{d}{db}\begin{bmatrix}\delta\tilde{r}\\ \delta\tilde{\lambda}\end{bmatrix} = \begin{bmatrix} 2-\frac{\epsilon}{3} & 3TS_{d-1}\left(1+\frac{\epsilon}{3(1-\epsilon)}\right) \\ 0 & -\epsilon \end{bmatrix}\begin{bmatrix}\delta\tilde{r}\\ \delta\tilde{\lambda}\end{bmatrix}} \quad (85)$$

eigenvalues $\boxed{-\epsilon}$ (86) and $\boxed{2-\frac{\epsilon}{3}}$ (87)

The $-\epsilon$ eigenvalue corresponds to $\delta\tilde{\lambda}$, which is now irrelevant

The $2-\frac{\epsilon}{3}$ corresponds to $\delta\tilde{r}$. This is still relevant, but the eigenvalue is now different from the mean-field value of 2. $\tilde{r}^*, \tilde{\lambda}^*$ is the scale invariant point where the correlation length is ∞.

A deviation will make ξ finite. To find the exponent ν, let us recall, from the notes on scaling relations

$$\boxed{2-\frac{\epsilon}{3} = \frac{d}{x} = 2-\alpha} \quad (88)$$

Eq (24) in "Scaling Relations"

$$\Rightarrow \quad \boxed{\alpha = \frac{\epsilon}{3}} \quad (89)$$

and hyperscaling $\boxed{\alpha = 2-\nu d}$ (90) Eq (51) in "Scaling Relations"

$$\boxed{\nu = \frac{2-\alpha}{d} = \frac{1}{x} = \frac{1}{2-\frac{\epsilon}{3}}} \quad (91)$$

For $\epsilon = 1$, or $d=3$ we get

$$\boxed{\nu = \frac{3}{5} = 0\cdot 6} \quad (92)$$

The experimental value is $\boxed{\nu_{exp} = 0\cdot 57.}$ (93)

For $\epsilon = 2$ or $d=2$ we get

$\nu = \frac{3}{4}$ compared to the correct value of $\nu = 1$ in 2D.

While the Wilsonian RG is great for understanding conceptually how RG works, and for computing exponents to order ϵ it is not convenient for going to higher orders. For this we turn to the Field-Theoretic RG.