Field Theoretic RG

From Wilsonian RG we learnt some important facts

1) High momenta correspond to short-distance fluctuations. Integrating them out and rescaling $\bar{k}, \bar{x}, \phi$ is a good way to do RG.

2) ϕ^6 and higher powers are irrelevant at tree level, as are higher derivatives, so for Ising Models we can confine ourselves to

$$\boxed{S = \int d^4x \left\{ \frac{1}{2}(\bar{\nabla}\phi)^2 + \frac{r}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \right\}} \quad (1)$$

3) Long wavelength correlations should be identical in the original and renormalized theories

So, we start with a path integral defined at cutoff Λ, with bare couplings r, λ, and calculate correlation functions at long wavelengths. In the process, we will need to rescale the fields (because the coefficient of $\frac{1}{2}(\bar{\nabla}\phi)^2$ is kept constant as the RG progresses). At the end of the day we may get a "renormalized" r, which I will call "m^2", and a renormalized dimensionless 4-pt. coupling, which I will call u.

The idea of Field-Theoretic RG is to keep the physical quantities (m^2 and u) fixed while increasing Λ. This requires tuning the bare couplings r, λ so that they now become functions of Λ
$r(\Lambda)$, $\lambda(\Lambda)$

The key point is this: The correlation length $\frac{1}{m}$ in "lattice" units $a \approx \frac{1}{\Lambda}$ is

$$\frac{\xi}{a} \simeq \frac{\Lambda}{m}$$

This diverges as $\Lambda \to \infty$ with m fixed.

So, any field theory with m fixed and $\Lambda \to \infty$ is close to criticality!!

This is the reason why all the techniques RG used in critical phenomena are so useful in QFT.

Furthermore, this leads us to the conclusion that the high-energy and low-energy theories can be quite different
Every field theory is an effective theory

Back to the parameters m^2 and u.

To define these precisely, we need the Irreducible Vertex functions. These can be obtained via a Legendre transformation of the Free energy F.

Let us add a "source" term, which in stat mech is simply a controllable local Zeeman field on the spin

$$\boxed{S[\phi] \rightarrow S[\phi, J] = \int d^d x \left\{ \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{r}{2}\phi^2 + \frac{\lambda}{4}\phi^4 - J(x)\phi(\vec{x}) \right\}} \quad (2)$$

$$\boxed{Z[J] = e^{-F[J]} = \int \mathcal{D}\phi \, e^{-S[\phi, J]}} \quad (3)$$

We already know that $F[J]$ is the generating functional of all correlators of ϕ.

Consider $-\frac{\delta F[J]}{\delta J(\vec{x})}$, which means vary the value of J only at one lattice site $\vec{x}$

$$\boxed{-\frac{\delta F[J]}{\delta J(\vec{x})} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(\vec{x})} = \langle \phi(\vec{x}) \rangle} \quad (4)$$

We will not set $J(\vec{x}) = 0$, so $\langle \phi(\vec{x}) \rangle$ is a functional of J

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let us call this $\boxed{\phi_{cl}(\bar{x};[J]) = -\dfrac{\delta F[J]}{\delta J(\bar{x})}}$

ϕ_{cl} is the "classical" field.
This is the magnetization at $\vec{x}$ in the presence of all the fields $J(\bar{x}')$ at all sites $\bar{x}'$

Now define

$$\boxed{\Gamma[\phi_{cl}] = F[J] + \int d^q x \; J(\bar{x})\,\phi_{cl}(\bar{x})} \qquad (6)$$

By the standard properties of Legendre transformation Γ is a functional of ϕ_{cl} and not of J, because

$$\boxed{\delta\Gamma[\phi_{cl}] = \int d^q x \left\{ \underbrace{\frac{\delta F[J]}{\delta J(\bar{x})}\,\delta J(\bar{x}) + \left(\delta J(\bar{x})\right)\phi_{cl}(\bar{x})}_{\text{zero by definition}} + J(\bar{x})\,\delta\phi_{cl}(\bar{x}) \right\}} \qquad (7)$$

Thus $\boxed{\dfrac{\delta\Gamma[\phi_{cl}]}{\delta\phi_{cl}(\bar{x})} = J(\bar{x})}$ (8)

Now expand Γ in powers of ϕ_{cl}. The different terms are irreducible vertex functions

$$\boxed{\Gamma[\phi_{cl}] = \Gamma[0] + \int d^q x\, \delta\phi_{cl}(\bar{x})\,\Gamma_1(\bar{x}) + \frac{1}{2!}\int d^q x_1\, d^q x_2\, \phi_{cl}(\bar{x}_1)\phi_{cl}(\bar{x}_2)\,\Gamma_2(\bar{x}_1,\bar{x}_2) + \cdots + \frac{1}{n!}\int d^q x_1 \cdots d^q x_n\; \phi_{cl}(\bar{x}_1)\cdots\phi_{cl}(\bar{x}_n)\,\Gamma_n(\bar{x}_1,\cdots \bar{x}_n)} \qquad (9)$$

To see the meaning of these vertex functions look at Γ_2.

$$\boxed{\begin{aligned}\Gamma_2(\bar{x}_1,\bar{x}_2) &= \frac{\delta^2\Gamma[\phi_{cl}]}{\delta\phi_{cl}(\bar{x}_2)\,\delta\phi_{cl}(\bar{x}_1)} = \frac{\delta J(\bar{x}_1)}{\delta\phi_{cl}(\bar{x}_2)}\\ &= \left\{\frac{\delta\phi_{cl}(\bar{x}_2)}{\delta J(\bar{x}_1)}\right\}^{-1} = \left\{-\frac{\delta^2 F[J]}{\delta J(\bar{x}_2)\,\delta J(\bar{x}_1)}\right\}^{-1}\\ &= \left\{G_2(\bar{x}_1-\bar{x}_2)\right\}^{-1} \qquad (10)\end{aligned}}$$

where $G_2(\bar{x}_1-\bar{x}_2) = \langle\phi(\bar{x}_1)\,\phi(\bar{x}_2)\rangle$

In $\bar{k}$-space the relation is even simpler

$$\boxed{\begin{aligned}G_2(\vec{k}) &= \frac{1}{k^2+r}\\ \Gamma_2(\bar{k}) &= k^2+r\end{aligned}} \quad (11) \qquad (\text{if } \lambda=0)$$

How about Γ_4? This is more complicated

$$\boxed{\begin{aligned}\Gamma_4(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4) &= \frac{\delta^3}{\delta\phi_{cl}(\bar{x}_2)\,\delta\phi_{cl}(\bar{x}_3)\,\delta\phi_{cl}(\bar{x}_4)}\left(J(\bar{x}_1)\right)\\ &= \frac{\delta^2}{\delta\phi_{cl}(\bar{x}_3)\,\delta\phi_{cl}(\bar{x}_4)}\left\{\frac{\delta^2 \ln Z[J]}{\delta J(\bar{x}_1)\,\delta J(\bar{x}_2)}\right\}^{-1} \qquad (12)\end{aligned}}$$

Now change variables

$$\boxed{\frac{\delta}{\delta \phi_{cl}(\bar{x}_3)} = \int d^d y_3 \, \frac{\delta J(\bar{y}_3)}{\delta \phi_{cl}(\bar{x}_3)} \, \frac{\delta}{\delta J(\bar{y}_3)}} \quad (13)$$

$$\Rightarrow \boxed{\Gamma_4 = \int d^d y_3 \, d^d y_4 \, \frac{\delta J(\bar{y}_4)}{\delta \phi_{cl}(\bar{x}_4)} \, \frac{\delta}{\delta J(\bar{y}_4)} \left\{ \frac{\delta J(\bar{y}_3)}{\delta \phi_{cl}(\bar{x}_3)} \, \frac{\delta}{\delta J(\bar{y}_3)} \left(\frac{\delta^2 \ln Z[J]}{\delta J(\bar{x}_1)\, \delta J(\bar{x}_2)} \right)^{-1} \right\}} \quad (14)$$

Now one must recall that the inverse is a matrix inverse, not a simple $\frac{1}{(\,)}$

Start with $\boxed{\mathbb{A}\, \mathbb{A}^{-1} = \mathbb{1}}$ (15)

$$\Rightarrow \boxed{(d\mathbb{A})\, \mathbb{A}^{-1} + \mathbb{A}\left(d\, \mathbb{A}^{-1}\right) = 0} \quad (16)$$

$$\Rightarrow \mathbb{A}\,(d\,\mathbb{A}^{-1}) = -(d\mathbb{A})\, \mathbb{A}^{-1}$$

Left-multiply by $\mathbb{A}^{-1}$

$$\boxed{d\mathbb{A}^{-1} = -\mathbb{A}^{-1}(d\mathbb{A})\, \mathbb{A}^{-1}} \quad (17)$$

$$\boxed{\begin{aligned} &\frac{\delta}{\delta J(\bar{y}_3)} \left\{ \frac{\delta^2 \ln Z[J]}{\delta J(\bar{x}_1)\, \delta J(\bar{x}_2)} \right\}^{-1} \\ &= \int d^d y_1 \, d^d y_2 \left\{ \frac{\delta^2 \ln Z[J]}{\delta J(\bar{x}_1)\, \delta J(\bar{y}_1)} \right\}^{-1} \left\{ \frac{\delta^3 Z[J]}{\delta J(\bar{y}_3)\delta J(\bar{y}_1)\, \delta J(\bar{y}_2)} \right\} \left\{ \frac{\delta^2 \ln Z[J]}{\delta J(\bar{y}_2)\delta J(\bar{x}_2)} \right\}^{-1} \end{aligned}} \quad (18)$$

At the end we will set $J=0$. Going back to (14), the outermost derivative $\frac{\delta}{\delta J(\bar{y}_4)}$ can in principle act on $\frac{\delta J(\bar{y}_3)}{\delta \phi_{cl}(\bar{x}_3)}$ as well as on any of the factors of (18).

However, when we set $J(\bar{x})=0$ at the end, all correlators with an odd number of ϕ's will vanish by symmetry. So the only nonvanishing term is when the $\frac{\delta}{\delta J(\bar{y}_4)}$ acts on the central factor of (18).

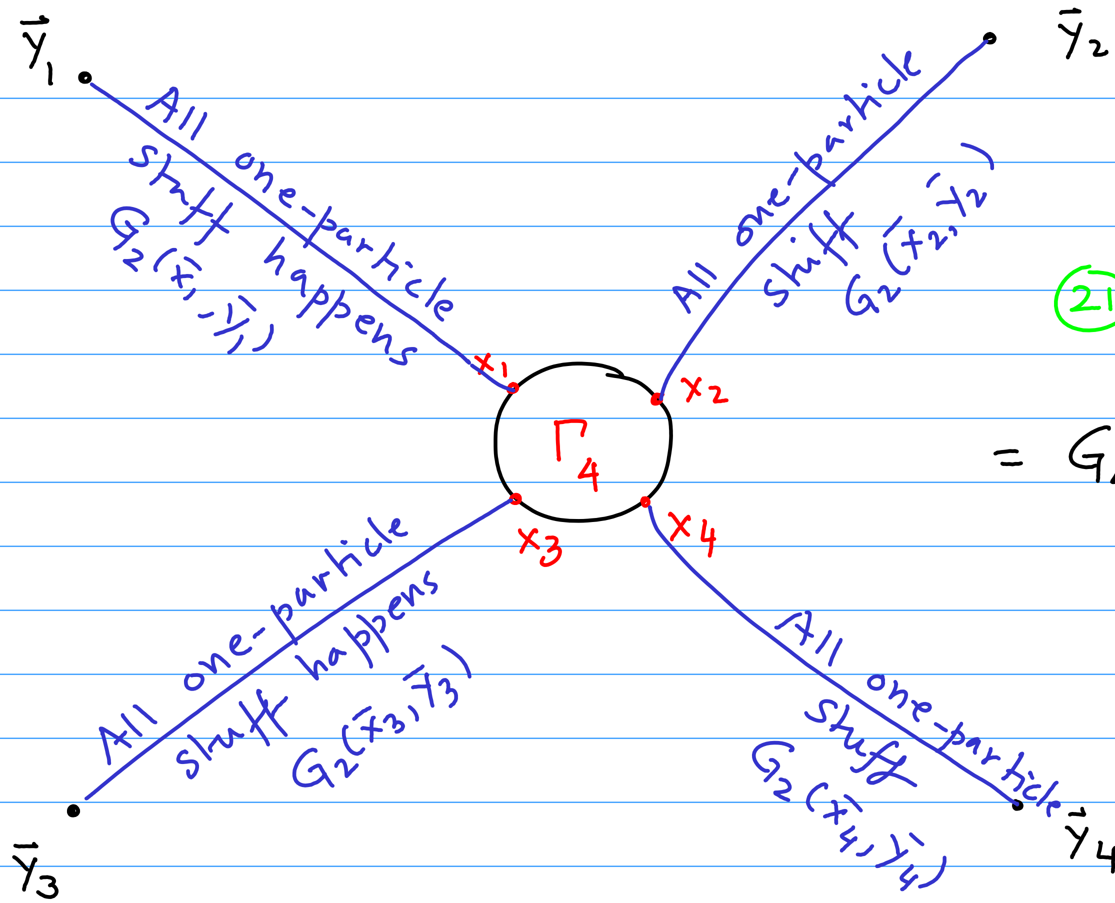
Thus

$$\boxed{\Gamma_4(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4) = \int d^d y_1\, d^d y_2\, d^d y_3\, d^d y_4 \, [G_2(\bar{x}_1,\bar{y}_1)]^{-1}[G_2(\bar{x}_2,\bar{y}_2)]^{-1}[G_2(\bar{x}_3,\bar{y}_3)]^{-1}[G_2(\bar{x}_4,\bar{y}_4)]^{-1} \left.\frac{\delta^4 \ln Z[J]}{\delta J(\bar{y}_1)\,\delta J(\bar{y}_2)\,\delta J(\bar{y}_3)\,\delta J(\bar{y}_4)}\right|_{J=0}} \quad (19)$$

This last factor is

$$\boxed{\left.\frac{\delta^4 \ln Z[J]}{\delta J(\bar{y}_1)\,\delta J(\bar{y}_2)\,\delta J(\bar{y}_3)\,\delta J(\bar{y}_4)}\right|_{J=0} = \langle \phi(\bar{y}_1)\phi(\bar{y}_2)\phi(\bar{y}_3)\phi(\bar{y}_4)\rangle_c = G_4(\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{y}_4)} \quad (20)$$

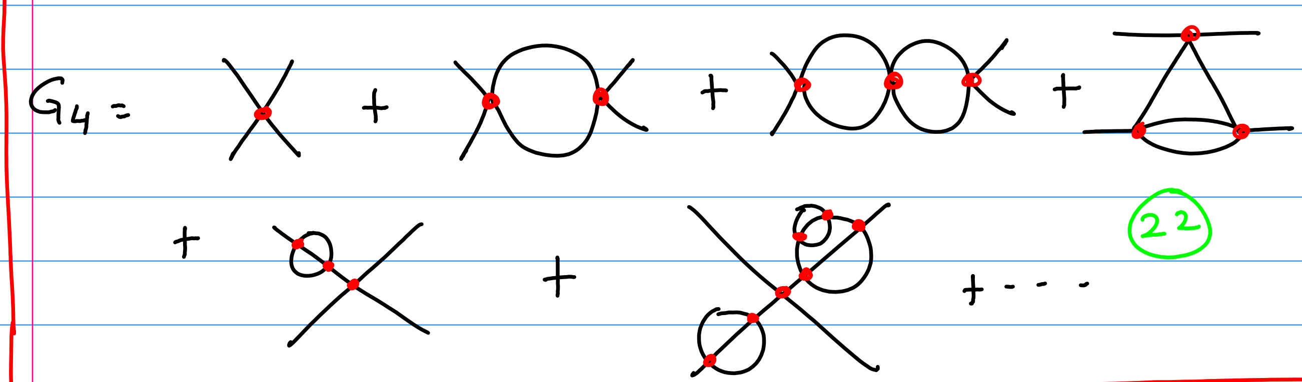
Here is a picture of the connection between Γ_4 and G_4



$$= G_4(\vec{y}_1,\vec{y}_2,\vec{y}_3,\vec{y}_4) \qquad (21)$$

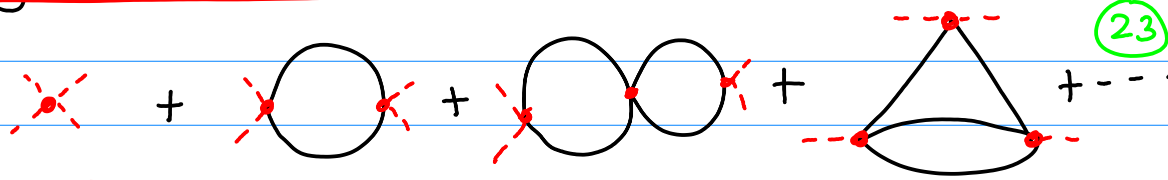
The process of going from G_4 to Γ_4 is sometimes called amputating the legs.

Γ_4 is the full 4-pt vertex. Diagrammatically



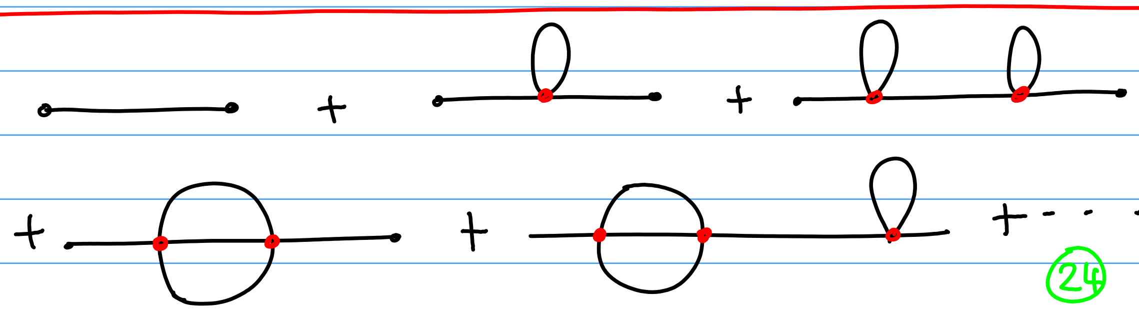
(22)

where the 1st line is the "dressing" of the vertex and the 2nd is the dressing of G_2 (self-energy). Only the vertex dressing enters Γ_4

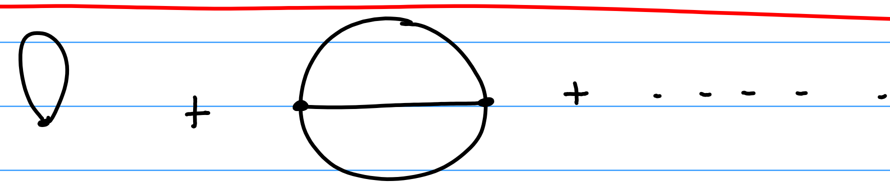
$\Gamma_4 =$  (23)

the - - - indicate "slots" where momenta attach.

Going back to G_2 and Γ_2 we can similarly express the relationship diagrammatic-ally

$G_2 =$  (24)

This is reorganized as follows: Take all diagrams which cannot be cut into two disjoint parts by cutting a single propagator and write them 1st as the self-energy $\Sigma(\bar{k})$

$\Sigma(\bar{k}) =$  (25)

Now one can write G_2 in terms of

$G_{20}(\bar{k}) =$  (26) as

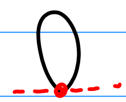
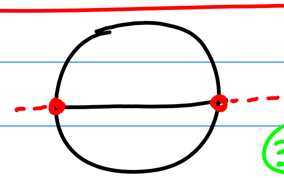
$$G_2(\bar{k}) = G_{20}(\bar{k}) + G_{20}(\bar{k})\,\Sigma(\bar{k})\,G_{20}(\bar{k}) + G_{20}\,\Sigma\,G_{20}\,\Sigma\,G_{20} + \cdots \quad (27)$$

This is easily recognized as a geometric series. Sum it up to obtain

$$\boxed{G_2(\bar{k}) = \frac{G_{20}(\bar{k})}{1 - \Sigma(\bar{k})\, G_{20}(\bar{k})} = \frac{1}{G_{20}^{-1}(\bar{k}) - \Sigma}} \quad (28)$$

From (10) we then infer

$$\boxed{\Gamma_2(\bar{k}) = G_{20}^{-1}(\bar{k}) - \Sigma} \quad (29)$$

$\Gamma_2(\bar{k}) = k^2 + r -$  $-$  (30)

Γ_2 consists of all the one-particle-irreducible diagrams of G_2, those that cannot be cut into two disjoint pieces by breaking one line (the free propagator)

Recall that in the quadratic theory

$$\Gamma_{20}(\bar{k}) = k^2 + r$$

The dressing will involve renormalizations of both the k^2 term and r

We rescale fields

$$\boxed{\phi_R(\bar{x}) = Z_\phi^{-\frac{1}{2}}\, \phi(\bar{x})} \quad (31)$$

in order to make the coefficient of the k^2 term in Γ_2 unity.

$$\boxed{\begin{aligned}&\int d^dx_1\, d^dx_2\; \Gamma_2(\bar{x}_1,\bar{x}_2)\, \phi_{cl}(\bar{x}_1)\phi_{cl}(\bar{x}_2) \qquad (32)\\ &= \int d^dx_1\, d^dx_2\; \phi_R(\bar{x}_1)\, \phi_R(\bar{x}_2)\; \Gamma_{2,R}(\bar{x}_1,\bar{x}_2)\end{aligned}}$$

$$\boxed{\Gamma_{2,R} = Z_\phi\, \Gamma_2} \quad (33)$$

So, the renormalization conditions on $\Gamma_{2,R}$ are

$$\boxed{\Gamma_{2,R}(\bar{k}=0) = m^2} \quad (34)$$

The physically observed mass

$$\boxed{\left.\frac{\partial \Gamma_{2,R}}{\partial k^2}\right|_{\bar{k}=0} = 1} \quad (35)$$

We will also need the physically observed 4-pt. coupling u

$$\boxed{\Gamma_{4,R} = Z_\phi^2\, \Gamma_4} \quad (36)$$

is a function of 3 momenta

and $$\boxed{\left.\Gamma_{4,R}\right|_{\bar{k}_i=0} = u\, m^{4-d}} \quad (37)$$

designed to make u dimensionless

$\Gamma_{2,R}$ and $\Gamma_{4,R}$ are independent of Λ

The field theoretic philosophy is the following: We know that we observe a mass^2 of m^2 and a 4-pt coupling of u at long-wavelengths.

What is the microscopic theory, OF THE SAME FORM, which leads to the given values of m^2, u?

In other words, we look for $r(\Lambda)$, $\lambda(\Lambda)$ such that we obtain m^2, u.

The motivation behind this line of reasoning is QED. We observe some mass for the electron, and a fine structure constant

$$\boxed{\alpha = \frac{e^2}{\hbar c}} \quad (38)$$

Physically we dont notice any cutoff. So we look for a theory as $\Lambda \to \infty$, with couplings $\tilde{m}(\Lambda)$ and $\tilde{\alpha}(\Lambda)$ such that at low energies we get the observed values of m, α.

This leads to a flow of $\tilde{m}(\Lambda)$ and $\tilde{\alpha}(\Lambda)$ as $\Lambda \to \infty$. This is a UV Flow.

In condensed matter, we ask the opposite question. Given a microscopic $r(\Lambda), \lambda(\Lambda)$ what m^2, u do we observe at long wavelengths?

In particular, we are interested in the critical point, $m^2=0$. To deal with this we need to modify the renormalization conditions.

With $m^2=0$, there is no intrinsic scale. We introduce a scale μ^2 at which we will make measurements.

So, the conditions on $\Gamma_{2,R}$ become

$$\boxed{\Gamma_{2,R}(\vec{k}=0)=0} \quad (39)$$

$$\boxed{\left.\frac{\partial \Gamma_{2,R}}{\partial k^2}\right|_{\vec{k}^2=\mu^2} = 1} \quad (40)$$

$\Gamma_{4,R}$ requires a bit more thought. Typically, we use the symmetric point (SP) in 4 dimensions

$$\boxed{\vec{k}_1^{\,2} = \vec{k}_2^{\,2} = \vec{k}_3^{\,2} = \mu^2} \quad (41)$$

but also $\quad k_4^2 = (\bar{k}_1+\bar{k}_2+\bar{k}_3)^2 = \mu^2$

$$\Rightarrow \quad 3\mu^2 + 2(\bar{k}_1\cdot\bar{k}_2 + \bar{k}_2\cdot\bar{k}_3 + \bar{k}_3\cdot\vec{k}_1) = \mu^2$$

$$\Rightarrow \quad \boxed{\bar{k}_i \cdot \bar{k}_j = -\frac{\mu^2}{3} \qquad i \neq j} \quad (42)$$

$$\Rightarrow \quad \boxed{\Gamma_{4,R}(\text{symm.pt}) = u\,\mu^{4-d}} \quad (43)$$

to make u dim.less

So, in the massless case we have

$$\boxed{\Gamma_{2R}(\bar{k}) = Z_\phi \; \Gamma_2(\bar{k})}$$ Repeat of 33

$$\boxed{\Gamma_{4R}(\{\bar{k}_i\}) = Z_\phi^2 \; \Gamma_4(\bar{k})}$$ Repeat of 36

and the conditions

$$\boxed{\Gamma_{2R}(\bar{k}=0) = 0}$$ Repeat of 39

$$\boxed{\left.\frac{\partial \Gamma_{2R}}{\partial k^2}\right|_{k^2=\mu^2} = 1}$$ Repeat of 40

$$\boxed{\Gamma_{4R}(\{\bar{k}_i\} = SP) = u\,\mu^{\epsilon}}$$ Repeat 43

Now since there is scale invariance at long wavelengths, we should find that

$$\boxed{G_{2R}(\bar{x}) \sim \frac{1}{|\bar{x}|^{d-2+\eta}}} \quad (44)$$

$$\Rightarrow \quad \boxed{G_{2R}(\bar{k}) \sim \int d^dx \, \frac{e^{-i\bar{k}\cdot\bar{x}}}{|\bar{x}|^{d-2+\eta}} = k^{\eta-2}} \quad (45)$$

$$\Rightarrow \quad \boxed{\Gamma_{2R}(\bar{k}) \sim k^{2-\eta} \quad \text{as } |\bar{k}| \to 0} \quad (46)$$

So we should be able to find η from

$$\boxed{2-\eta = \left.\frac{\partial \Gamma_{2R}}{\partial \ln k}\right|_{k\to 0}} \quad (47)$$

Of course, we know generally that in order to have scale invariance we should be at a fixed pt. So u should be a scale-independent constant.

In this version of RG we want to 1st find the β-f^n of u

$$\boxed{\beta_u = -\mu \frac{\partial u}{\partial \mu}} \tag{48}$$

minus because μ is a mass scale or a momentum scale

How to find β_u ?

Let us go back to the defining eq^{n}s and ask what the possible dependences of various quantities are.

The bare coupling λ has scaling dimension ϵ and thus

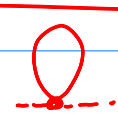
$$\boxed{\lambda = \tilde{\lambda}\,\Lambda^{\epsilon}} \tag{49}$$

where $\tilde{\lambda}$ is dimensionless. Γ_4 has a perturbative expansion in $\tilde{\lambda}$

$$\boxed{\Gamma_4(\{\bar{k}_i\}) = \Lambda^{\epsilon}\left\{\tilde{\lambda} + F_4\,\tilde{\lambda}^2 I_4(\{\bar{k}_i\}) + \cdots\right\}} \tag{50}$$

where F_4 is a numerical factor

We already know that in Γ_2, the renormalization of the k^2 term occurs at order $\tilde{\lambda}^2$

$$\Rightarrow \quad \Gamma_2(\bar{k}) = k^2 + r + \tilde{\lambda}$$

$$+ F_2\,\tilde{\lambda}^2 I_2(\bar{k}) + \cdots \tag{51}$$

We want to isolate the part of I_2 which has $\bar{k}$-dependence

$$\Gamma_2(\bar{k}) = k^2 + r + \tilde{\lambda} \; + F_2 \tilde{\lambda}^2 I_2(0) + F_2 \tilde{\lambda}^2 \left(I_2(\bar{k}) - I_2(0) \right) + \cdots \tag{52}$$

Now $r + \tilde{\lambda} \; + F_2 \tilde{\lambda}^2 I_2(0) = m^2$ and in the scale invariant case $m^2 = 0$ (53)

$$\Gamma_2(\bar{k}) = k^2 + F_2 \tilde{\lambda}^2 \left(I_2(\bar{k}) - I_2(0) \right\} \tag{54}$$

$$\frac{\partial \Gamma_2(\bar{k})}{\partial k^2} = 1 + F_2 \tilde{\lambda}^2 \frac{\partial I_2(\bar{k})}{\partial k^2}$$

From (33), (40) we infer that in perturbation theory

$$Z_\phi = 1 - F_2 \tilde{\lambda}^2 \left.\frac{\partial I_2}{\partial k^2}\right|_{k^2 = \mu^2} + \cdots \tag{55}$$

From (36), (43) we know

$$\Gamma_{4R}(\{\bar{k}_i\} = SP) = u\mu^\epsilon = Z_\phi^2 \, \Gamma_4(\{\bar{k}_i\} = SP)$$

$$u\mu^\epsilon = \left\{ 1 - 2F_2 \tilde{\lambda}^2 \left.\frac{\partial I_2}{\partial k^2}\right|_{k^2 = \mu^2} + \cdots \right\} \Lambda^\epsilon \left\{ \tilde{\lambda} + F_4 \tilde{\lambda}^2 I_4 \Big|_{SP} + \cdots \right\}$$

$$= \Lambda^\epsilon \left\{ \tilde{\lambda} + F_4 \tilde{\lambda}^2 I_4 \Big|_{SP} + \cdots \right\} \tag{56}$$

$$\Rightarrow \quad \boxed{u = \left(\frac{\Lambda}{\mu}\right)^{\epsilon} \left\{ \tilde{\lambda} + F_4 \tilde{\lambda}^2 I_4 \Big|_{SP} + \cdots \right\}} \qquad (57)$$

Invert this perturbatively to obtain

$$\boxed{\tilde{\lambda} = \left(\frac{\mu}{\Lambda}\right)^{\epsilon} \left\{ u - F_4 \left(\frac{\mu}{\Lambda}\right)^{\epsilon} u^2 I_4 \Big|_{SP} + \cdots \right\}} \qquad (58)$$

Now the key point is that the LHS knows nothing about the renormalization scale μ

$$\Rightarrow \quad -\mu \frac{d\tilde{\lambda}}{d\mu} = 0 = \left(\frac{\mu}{\Lambda}\right)^{\epsilon} \left\{ 1 - 2F_4 \left(\frac{\mu}{\Lambda}\right)^{\epsilon} u\, I_4 \Big|_{SP} + \cdots \right\} \beta_u \qquad (59)$$

$$- \epsilon \left(\frac{\mu}{\Lambda}\right)^{\epsilon} u + u^2 F_4\, \mu \frac{\partial}{\partial \mu} \left(\left(\frac{\mu}{\Lambda}\right)^{\epsilon} I_4 \Big|_{SP} \right) + \cdots$$

From which we can deduce β_u

$$\beta_u = \epsilon u - u^2 F_4 \left(\frac{\Lambda}{\mu}\right)^{\epsilon} \mu \frac{\partial}{\partial \mu} \left[\left(\frac{\mu}{\Lambda}\right)^{\epsilon} I_4 \Big|_{SP} \right] \qquad (60)$$

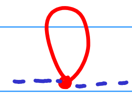
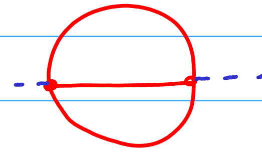
$$+ 2F_4 u^2 \left(\frac{\mu}{\Lambda}\right)^{\epsilon} I_4 \Big|_{SP} + \cdots$$

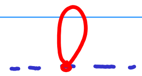
Now find the fixed point u^* by setting $\beta_u = 0$. From (54), (55)

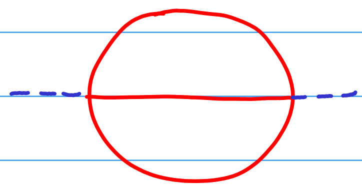
$$\frac{\partial \Gamma_{2R}}{\partial \ln k} = \left[1 - F_2 \left(\frac{\mu}{\Lambda}\right)^{2\epsilon} u^2 \frac{\partial I_2(k)}{\partial k^2} \Big|_{\mu^2} + \cdots \right] \left[1 + F_2 \left(\frac{\mu}{\Lambda}\right)^{2\epsilon} u^2 \frac{\partial I_2}{\partial \ln k} + \cdots \right] \qquad (61)$$

$$\boxed{= 2 - \eta} \quad \text{at} \quad u = u^*$$

Let us proceed to do some real calculation!
First we want to find $\Gamma_2(\bar{k})$. From (30)
this is

$\Gamma_2(\bar{k}) = k^2 + r$ −  − 

The  term comes from $\langle V \rangle_0$
in the linked cluster expansion, while

 comes from $-\frac{1}{2}\langle V^2 \rangle_{0c}$

There are numerical factors appearing with
these diagrams. To figure them out let us
compute $G_2(\bar{k})$ and from there obtain $\Gamma_2(\bar{k})$

$$\boxed{\begin{aligned}\langle \phi(\bar{k})\,\phi(\bar{k}')\rangle_c &= (2\pi)^d\,\delta^d(\bar{k}+\bar{k}')\,G_2(\bar{k}) \\ &= \frac{1}{Z}\int \mathcal{D}\phi \;\; \phi(\bar{k})\,\phi(\bar{k}')\, e^{-S_0 - V}\end{aligned}} \quad (62)$$

$$\boxed{S_0 = \int \frac{d^dk}{(2\pi)^d}\, |\phi(\bar{k})|^2 (\bar{k}^2 + r) \quad (63)} \Rightarrow \boxed{\langle \phi(\bar{k})\,\phi(\bar{k}')\rangle_0 = \frac{(2\pi)^d\,\delta^d(\bar{k}+\bar{k}')}{k^2+r} \quad (64)}$$

$$\boxed{V = \frac{\lambda}{4}\int \frac{d^dk_1\, d^dk_2\, d^dk_3\, d^dk_4}{(2\pi)^{3d}}\, \delta^d\Big(\sum_i k_i\Big)\,\phi(\bar{k}_1)\,\phi(\bar{k}_2)\,\phi(\bar{k}_3)\,\phi(\bar{k}_4) \quad (65)}$$

To 1^{st} order we obtain

$$-\frac{\lambda}{4}\int \frac{d^d k_1 \cdots d^d k_4}{(2\pi)^{3d}}\, \delta^d\Big(\sum_i \vec{k}_i\Big) \langle \phi(\vec{k})\, \phi(\vec{k}')\, \phi(\bar{k}_1)\, \phi(\bar{k}_2)\, \phi(\bar{k}_3)\, \phi(\bar{k}_4) \rangle \qquad 66$$

1^{st} choose a ϕ from the $\lambda\phi^4$ to contract with $\phi(\bar{k})$. This can be done in 4 ways. Next choose another ϕ from $\lambda\phi^4$ to contract with $\phi(\bar{k}')$. This can be done in 3 ways

$\Rightarrow$ We get

$$-3\lambda \int \frac{d^d k_1 \cdots d^d k_4}{(2\pi)^{3d}}\, \delta^d\Big(\sum_i \bar{k}_i\Big) \langle \phi(\bar{k})\, \phi(\bar{k}_1) \rangle_0 \langle \phi(\bar{k}')\, \phi(\bar{k}_2) \rangle_0 \langle \phi(\bar{k}_3)\, \phi(\bar{k}_4) \rangle_0 \qquad 67$$

Plug in 41 and simplify to get the 1^{st} order contribution as

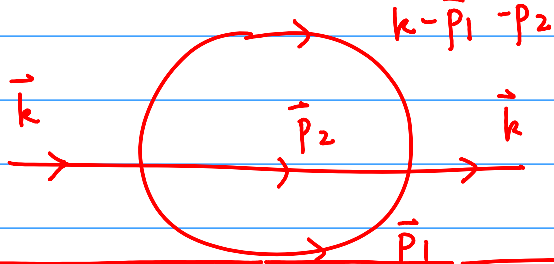
$$-3\lambda\, (2\pi)^d\, \delta^d(\bar{k}+\bar{k}')\, G_{20}(\bar{k}) \left\{ \int_0^\Lambda \frac{d^d k_3}{(2\pi)^d} \frac{1}{k_3^2 + r} \right\} G_{20}(\bar{k}) \qquad 68$$

Now consider the 2^{nd} order. There is a diagram

69

This is one-particle-reducible, which means it can be cut into two disjoint pieces upon cutting one propagator. So it is not a part of $\Sigma(\bar{k})$.

To 2^{nd} order the only 1PI (one-particle-irreducible) diagram is



(70)

$$\boxed{= \frac{1}{2}\left(\frac{\lambda}{4}\right)^2 \int \frac{d^dk_1 \cdots d^dk_8}{(2\pi)^{6d}} \delta^d(\bar{k}_1+\bar{k}_2+\bar{k}_3+\bar{k}_4)\, \delta^d(\vec{k}_5+\vec{k}_6+\bar{k}_7+\bar{k}_8) \quad (71) \\ \langle \phi(\bar{k})\phi(\bar{k}')\, \phi(\bar{k}_1)\phi(\bar{k}_2)\phi(\bar{k}_3)\phi(\bar{k}_4)\, \phi(\bar{k}_5)\phi(\bar{k}_6)\phi(\bar{k}_7)\phi(\bar{k}_8)\rangle_0}$$

First pick a vertex to contract with $\phi(\bar{k})$. Since there are two choices there is a factor of 2. Next pick a ϕ from the $\lambda\phi^4$ of the 1^{st} vertex to contract with $\phi(\bar{k})$. This gives a factor of 4. Next pick a ϕ from the 2^{nd} vertex to contract with $\phi(\bar{k}')$, again getting a factor of 4. Now there are 3 ϕ's in the 1^{st} vertex which have to contract with 3 ϕ's of the second vertex, which can be done in $3! = 6$ ways. So, in all we get a factor of

$$\boxed{\frac{1}{2}\left(\frac{\lambda}{4}\right)^2 2 \times 4 \times 4 \times 6 = 6\lambda^2} \quad (72)$$

The rest is

$$\boxed{G_0(\bar{k})\, I_2(\bar{k})\, G_0(\bar{k})} \quad (73)$$

where $$\boxed{I_2(\bar{k}) = \int_0^{\Lambda} \frac{d^d\bar{p}_1\, d^d\bar{p}_2}{(2\pi)^{2d}} \frac{1}{(\bar{p}_1^2+r)(\bar{p}_2^2+r)((\bar{p}_1+\bar{p}_2-\bar{k})^2+r)}} \quad (74)$$

So, to 2nd order we have

$$\boxed{\Gamma_2(\bar{k}) = k^2 + r + 3\lambda \int_0^{\Lambda} \frac{d^d\bar{p}}{(2\pi)^d} \frac{1}{p^2+r} - 6\lambda^2 I_2(\bar{k})} \quad (75)$$

Now consider $G_4(\{\bar{k}_i\})$, $\Gamma_4(\{\bar{k}_i\})$ from (22) (23)

The 1st term in (22) comes with a factor $\boxed{-\frac{\lambda}{4} \cdot 4! = -6\lambda}$ (76) because of 4! the ways of attaching the external lines to the vertex.

In the 2nd term in (22) we start with $\frac{1}{2}\left(\frac{\lambda}{4}\right)^2$. Now choose a vertex to attach to $\bar{k}_1$, which gives a factor of 2.

Next, a factor of 4 to choose a ϕ in the 1st vertex to attach to $\phi(\bar{k}_1)$. Now choose another $\bar{k}_i$ among the remaining external legs to attach to the 1st vertex $\Rightarrow$ factor of 3

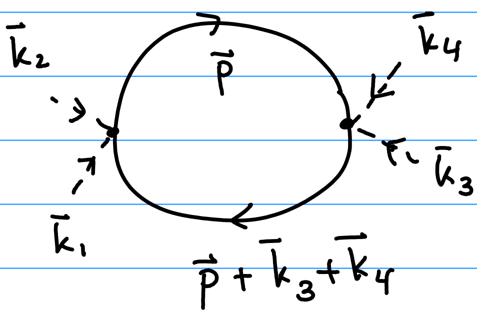
Now having chosen the external leg to attach to the 1st vertex, choose one of the remaining 3 vertex lines to attach it to $\Rightarrow$ factor of 3

The remaining 2 external legs attach to 2 of the lines in the 2nd vertex $\Rightarrow$ 12 ways
Finally, the 2 lines from the 1st vertex attach to the 2 lines from the 2nd in 2 ways

$$\Rightarrow \boxed{\begin{aligned} \text{factor} &= \frac{1}{2}\left(\frac{\lambda}{4}\right)^2 \; 2\times 12\times 3\times 12\times 2 \\ &= 54\lambda^2 \end{aligned}} \quad (77)$$

Extracting the common factor of -6 we get

$$\boxed{\Gamma_4(\{\vec{k}_i\}) = \lambda - 9\lambda^2 I_4(\{\vec{k}_i\}) + \cdots} \quad (78)$$

$I_4(\{\vec{k}_i\}) =$ 

Recall
$\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4=0$

$$\boxed{I_4(\{\vec{k}_i\}) = \int_0^{\Lambda} \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2+r)\left[(\vec{p}+\vec{k}_3+\vec{k}_4)^2+r\right]}} \quad (79)$$

To find β_u from (60) we need only I_4

From (37), (40) in the "Feynman trick" set of notes

$$\boxed{I_4(\{\bar{k}_i\}) = \frac{S_{d-1}}{(2\pi)^d}\,\frac{\Gamma(\frac{d}{2})\Gamma(2-\frac{d}{2})}{2}\,\frac{[\Gamma(1-\frac{\epsilon}{2})]^2}{\Gamma(2-\epsilon)}\,\frac{1}{[(\bar{k}_3+\bar{k}_4)^2]^{\epsilon/2}}} \quad (80)$$

$(\vec{k}_3+\vec{k}_4)^2 = k_3^2+k_4^2+2\bar{k}_3\cdot\bar{k}_4$. At the symmetric point

$$= \mu^2+\mu^2-\frac{2\mu^2}{3} = \frac{4\mu^2}{3}$$

Use $\boxed{\lambda = \tilde{\lambda}\,\Lambda^{\epsilon}}$ (81)

$$\boxed{\begin{aligned}\Gamma_4\Big|_{SP} &= \tilde{\lambda}\Lambda^{\epsilon} - \frac{\tilde{\lambda}^2\Lambda^{2\epsilon}}{(\frac{4}{3}\mu^2)^{\frac{\epsilon}{2}}}\,\frac{9S_{d-1}}{(2\pi)^d}\,\frac{\Gamma(\frac{d}{2})\Gamma(\frac{\epsilon}{2})[\Gamma(1-\frac{\epsilon}{2})]^2}{2\Gamma(2-\epsilon)} + \cdots \\ &= u\,\mu^{\epsilon}\end{aligned}} \quad (82)$$

$$\Rightarrow \boxed{u = \left(\frac{\Lambda}{\mu}\right)^{\epsilon}\left\{\tilde{\lambda} - \frac{9\tilde{\lambda}^2}{(\frac{4}{3})^{\epsilon/2}}\left(\frac{\Lambda}{\mu}\right)^{\epsilon}\frac{S_{d-1}}{(2\pi)^d}\,\frac{\Gamma(\frac{d}{2})\Gamma(\frac{\epsilon}{2})[\Gamma(1-\frac{\epsilon}{2})]^2}{2\Gamma(2-\epsilon)} + \cdots\right\}} \quad (83)$$

Invert

$$\boxed{\tilde{\lambda} = \left(\frac{\mu}{\Lambda}\right)^{\epsilon}\left\{u + \frac{9u^2}{(\frac{4}{3})^{\epsilon/2}}\,\frac{S_{d-1}}{(2\pi)^d}\,\frac{\Gamma(\frac{d}{2})\Gamma(\frac{\epsilon}{2})[\Gamma(1-\frac{\epsilon}{2})]^2}{2\Gamma(2-\epsilon)} + \cdots\right\}} \quad (84)$$

Let us simplify the ratio of Γ functions by assuming $\epsilon << 1$ $d \approx 4$

$$\Gamma(\tfrac{\epsilon}{2}) \approx \frac{2}{\epsilon} \qquad \Gamma(\tfrac{d}{2}) \approx \Gamma(2) = 1$$

$$\Gamma\left(1-\frac{\epsilon}{2}\right) \approx 1 \qquad \Gamma(2-\epsilon) \approx 1 \qquad \left(\frac{4}{3}\right)^{\epsilon/2} \approx 1$$

$$S_{d-1} = S_3 = 2\pi^2$$

$$\boxed{\begin{aligned} \tilde{\lambda} &= \left(\frac{\mu}{\Lambda}\right)^{\epsilon} \left\{ u + 9u^2 \frac{2\pi^2}{16\pi^4} \frac{1}{\epsilon} + \cdots \right\} \\ &= \left(\frac{\mu}{\Lambda}\right)^{\epsilon} \left\{ u + \frac{9u^2}{8\pi^2 \epsilon} + \cdots \right\} \end{aligned}} \tag{85}$$

$$\boxed{\begin{aligned} -\mu \frac{d\tilde{\lambda}}{d\mu} = 0 &= -\epsilon \left(\frac{\mu}{\Lambda}\right)^{\epsilon} \left\{ u + \frac{9u^2}{8\pi^2\epsilon} + \cdots \right\} \\ &\quad + \left(\frac{\mu}{\Lambda}\right)^{\epsilon} \beta_u \left\{ 1 + \frac{18u}{8\pi^2\epsilon} + \cdots \right\} \end{aligned}} \tag{86}$$

$$\Rightarrow \beta_u = \epsilon \left(1 - \frac{18u}{8\pi^2\epsilon} + \cdots\right)\left(u + \frac{9u^2}{8\pi^2\epsilon} + \cdots\right)$$

$$\approx \epsilon u - \frac{9u^2}{8\pi^2} + \cdots$$

to this order in u

$$\boxed{\beta_u(u) = \epsilon u - \frac{9u^2}{8\pi^2} + \cdots} \tag{87}$$

$$\Rightarrow \quad \boxed{u^* = \frac{8\pi^2\epsilon}{9}} \tag{88}$$

Fixed point

Now consider Γ_2

$$\boxed{\Gamma_2(\bar{k}) = k^2 + r + 3\lambda \int_0^\Lambda \frac{d^d\bar{p}}{(2\pi)^d} \frac{1}{p^2+r} - 6\lambda^2 I_2(\bar{k})} \quad (75)$$

Since $\Gamma_2(0)=0$ for the massless case

$$r + 3\lambda \int_0^\Lambda \frac{d^dp}{(2\pi)^d} \frac{1}{p^2+r} - 6\lambda^2 I_2(0) = 0$$

From Eq (55) of the "Feynman trick" notes

$$\boxed{\tilde{I}_2(\bar{k}) = \left[\frac{S_{d-1}}{(2\pi)^d}\right]^2 \frac{\Gamma(\frac{d}{2})\left[\Gamma(1-\frac{\epsilon}{2})\right]^3 \Gamma(-1+\epsilon)}{4\,\Gamma(3-\frac{3}{2}\epsilon)} (k^2)^{1-\epsilon}} \quad (55)$$

$$\boxed{\frac{\partial \Gamma_2}{\partial k^2} = 1 - 6(1-\epsilon)(k^2)^{-\epsilon}\lambda^2 \left[\frac{S_{d-1}}{(2\pi)^d}\right]^2 \frac{\Gamma(\frac{d}{2})\left[\Gamma(1-\frac{\epsilon}{2})\right]^3\Gamma(-1+\epsilon)}{4\Gamma(3-\frac{3}{2}\epsilon)}} \quad (89)$$

Now set $k^2=\mu^2$ and demand

$$\boxed{Z_\phi \left.\frac{\partial \Gamma_2}{\partial k^2}\right|_{k^2=\mu^2} = 1} \quad \text{Repeat of } (40)$$

Use (81), (85) $\lambda^2 (\mu^2)^{-\epsilon} = u^2$ to this order

$$\boxed{Z_\phi = 1 + 6(1-\epsilon)\,u^2 \left[\frac{S_{d-1}}{(2\pi)^d}\right]^2 \frac{\Gamma(\frac{d}{2})\left[\Gamma(1-\frac{\epsilon}{2})\right]^3 \Gamma(-1+\epsilon)}{4\,\Gamma(3-\frac{3}{2}\epsilon)}} \quad (90)$$

Now we $\epsilon << 1$

$$(-1+\epsilon)\,\Gamma(-1+\epsilon) = \Gamma(\epsilon) \simeq \frac{1}{\epsilon}$$

$$\Rightarrow \quad \Gamma(-1+\epsilon) \simeq -\frac{1}{\epsilon}$$

$$\Gamma(\tfrac{d}{2}) \simeq \Gamma(2) = 1 \qquad \Gamma(3-\tfrac{3}{2}\epsilon) \simeq \Gamma(3) = 2! = 2$$

$$\Gamma(1-\tfrac{\epsilon}{2}) \simeq \Gamma(1) = 1 \qquad S_{d-1} = 2\pi^2$$

$$\boxed{Z_\phi \simeq 1 - 6\,u^2 \left[\frac{1}{8\pi^2}\right]^2 \frac{1}{8\,\epsilon}} \quad (91)$$

Now we can write $\Gamma_{2,R}$ to this order

$$\Gamma_{2,R}(k) = \left\{1 - \frac{3u^2}{4\epsilon}\left[\frac{1}{8\pi^2}\right]^2 + \cdots\right\}\left\{k^2 + \frac{3u^2}{4\epsilon}\left(\frac{k^2}{\mu^2}\right)^{-\epsilon}\frac{k^2}{(8\pi^2)^2} + \cdots\right\}$$

$$\boxed{\Gamma_{2,R}(k) = k^2\left\{1 + \frac{3u^2}{4\epsilon}\left[\frac{1}{8\pi^2}\right]^2\left(\left(\frac{k^2}{\mu^2}\right)^{-\epsilon} - 1\right) + \cdots\right\}} \quad (92)$$

Now $$\boxed{\left(\frac{k^2}{\mu^2}\right)^{-\epsilon} = e^{-2\epsilon \ln\frac{k}{\mu}} \approx 1 - 2\epsilon \ln\frac{k}{\mu} + \cdots} \quad (93)$$

$$\Gamma_{2,R}(k) \simeq k^2\left\{1 + \frac{3u^2}{4}\left[\frac{1}{8\pi^2}\right]^2 \frac{1}{\epsilon}\left(-2\epsilon \ln\frac{k}{\mu}\right)\right\}$$

$$\boxed{\Gamma_{2,R}(k) = k^2 \left\{ 1 - 3\frac{u^2}{2}\left[\frac{1}{8\pi^2}\right]^2 \ln\frac{k}{\mu} + \cdots \right\}} \quad (94)$$

Now set $u = u^* = \frac{8\pi^2\epsilon}{9}$

$$\boxed{\Gamma_{2,R}(k)\Big|_{u^*} = k^2\left\{1 - \frac{\epsilon^2}{54}\ln\frac{k}{\mu} + \cdots\right\} \simeq k^{2-\frac{\epsilon^2}{6}}} \quad (95)$$

$$\Rightarrow \quad 2 - \eta = 2 - \frac{\epsilon^2}{54}$$

$\boxed{\eta = \frac{\epsilon^2}{54}}$ (96) is the leading order result

The actual value in $d=3$ is 0.04, which is twice the ϵ-expansion result.