The Feynman trick and some integrals

In doing loop integrals we often come across integrands of the form

$$\boxed{\frac{1}{(p_1^2+m^2)^{\nu_1}}\,\frac{1}{(p_2^2+m^2)^{\nu_2}}\cdots} \quad ①$$

where ν_1 and ν_2 are some powers and eventually the integral over $\bar{p}_1, \bar{p}_2 \cdots$ has to be performed. One example is the two-loop integral $I_2(\bar{k})$

$$\boxed{I_2(\bar{k}) = \int \frac{d^d\bar{p}_1 \; d^d\bar{p}_2}{(2\pi)^{2d}} \frac{1}{(p_1^2+m^2)(p_2^2+m^2)\left[(\bar{p}_1+\bar{p}_2-\bar{k})^2+m^2\right]}} \quad ②$$

Feynman invented a beautiful trick to simplify such integrations.

Consider a denominator D (usually $\bar{p}^2+m^2$) > 0

Consider $\boxed{\int_0^\infty d\alpha\, e^{-\alpha D}\, \alpha^{\nu-1}}$ ③ $\qquad \tilde{\alpha} = \alpha D$

$$= \frac{1}{D^\nu}\int_0^\infty d\tilde{\alpha}\, \tilde{\alpha}^{\nu-1} e^{-\tilde{\alpha}} = \frac{\Gamma(\nu)}{D^\nu}$$

$$\boxed{\frac{1}{D^\nu} = \frac{1}{\Gamma(\nu)}\int_0^\infty d\alpha\, \alpha^{\nu-1} e^{-\alpha D}} \quad ④$$

$$\Rightarrow \boxed{\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_M^{\nu_M}} = \frac{1}{\Gamma(\nu_1)\cdots\Gamma(\nu_M)} \int_0^\infty d\alpha_1\, \alpha_1^{\nu_1-1}\, d\alpha_2\, \alpha_2^{\nu_2-1} \cdots d\alpha_M\, \alpha_M^{\nu_M-1}\, e^{-(\alpha_1 D_1 + \alpha_2 D_2 + \cdots + \alpha_M D_M)}} \quad (5)$$

Now define $\boxed{t = \sum_1^M \alpha_i}$ (6)

and $\boxed{\tilde{\alpha}_i = \frac{\alpha_i}{t}}$ (7) Clearly $\boxed{\sum \tilde{\alpha}_i = 1}$ (8)

$$\boxed{\frac{1}{D_1^{\nu_1}\cdots D_M^{\nu_M}} = \frac{1}{\Gamma(\nu_1)\cdots\Gamma(\nu_M)} \int_0^\infty dt\ \delta\Big[t - \sum_1^M \alpha_i\Big]\, d\alpha_1\, \alpha_1^{\nu_1-1} \cdots d\alpha_M\, \alpha_M^{\nu_M-1}\, e^{-\sum_{i=1}^M \alpha_i D_i}} \quad (9)$$

$$\boxed{\delta[t - \sum \alpha_i] = \delta\Big[t - t\sum_i \tilde{\alpha}_i\Big] = \frac{1}{t}\,\delta[1 - \sum \tilde{\alpha}_i]} \quad (10)$$

Also $\boxed{d\alpha_i\, \alpha_i^{\nu_i-1} = t^{\nu_i}\, d\tilde{\alpha}_i\, \tilde{\alpha}_i^{\nu_i-1}}$ (11)

$$\boxed{\frac{1}{D_1^{\nu_1}\cdots D_M^{\nu_M}} = \frac{1}{\Gamma(\nu_1)\cdots\Gamma(\nu_M)} \int_0^\infty dt\ t^{\sum_i^M \nu_i - 1} \ \circledast \ \left\{\prod_{i=1}^M \int_0^1 d\tilde{\alpha}_i\ \tilde{\alpha}_i^{\nu_i-1}\right\} e^{-t\sum_{i=1}^M \tilde{\alpha}_i D_i}\ \delta\Big[1 - \sum_i^M \tilde{\alpha}_i\Big]} \quad (12)$$

For convenience define $\boxed{\nu = \sum_i^M \nu_i}$ (13)

Now we can do the t integral to get

$$\frac{1}{D_1^{\nu_1}\cdots D_M^{\nu_M}} = \frac{\Gamma(\nu)}{\left\{\prod_i \Gamma(\nu_i)\right\}} \int_0^1 \left\{\prod_{i=1}^{M} d\tilde{\alpha}_i\, \tilde{\alpha}_i^{\nu_i - 1}\right\} \frac{\delta\left[1-\sum_i^M \tilde{\alpha}_i\right]}{\left[\tilde{\alpha}_1 D_1 + \cdots + \tilde{\alpha}_M D_M\right]^{\nu}} \quad (14)$$

Let us try this on $I_2(\bar{k})$

$$\frac{1}{(p_1^2+m^2)(p_2^2+m^2)((\bar{p}_1+\bar{p}_2-\bar{k})^2+m^2)}$$

$$= \frac{\Gamma(3)}{[\Gamma(1)]^3} \int_0^1 \frac{d\tilde{\alpha}_1\, d\tilde{\alpha}_2\, d\tilde{\alpha}_3 \quad \delta[1-\tilde{\alpha}_1-\tilde{\alpha}_2-\tilde{\alpha}_3]}{\left[\tilde{\alpha}_1(p_1^2+m^2)+\tilde{\alpha}_2(p_2^2+m^2)+\tilde{\alpha}_3((\bar{p}_1+\bar{p}_2-\bar{k})^2+m^2)\right]^3}$$

$$= 2\int_0^1 d\tilde{\alpha}_1\, d\tilde{\alpha}_2 \frac{1}{\left[\tilde{\alpha}_1(p_1^2+m^2)+\tilde{\alpha}_2(p_2^2+m^2)+(1-\tilde{\alpha}_1-\tilde{\alpha}_2)((\bar{p}_1+\bar{p}_2-\bar{k})^2+m^2)\right]^3} \quad (15)$$

We still need to do the $d^d\bar{p}_1\, d^d\bar{p}_2$ integrals. We need a cutoff to eliminate UV problems. It will be convenient to be able to shift $\bar{p}_1$ and $\bar{p}_2$, so we choose a cutoff scheme that allows us to do this, the Pauli-Villars scheme. In this scheme we replace

$$\int_0^{\Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+m^2} \longrightarrow \int_0^{\infty} \frac{d^d p}{(2\pi)^d} \left\{\frac{1}{p^2+m^2} - \frac{1}{p^2+\Lambda^2}\right\} \quad (16)$$

$$I_2(\bar{k}) \longrightarrow \int \frac{d^d\bar{p}_1\, d^d\bar{p}_2}{(2\pi)^{2d}} \left\{\frac{1}{(p_1^2+m^2)(p_2^2+m^2)((\bar{p}_1+\bar{p}_2-\bar{k})^2+m^2)} - m\to\Lambda\right\} \quad (17)$$

In $d=4$ this is sometimes not enough. For example, in the 1^{st} integral there is still a log divergence. In that case the Pauli-Villars scheme is a bit more complicated.

$$\boxed{\int^{\Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+m^2} \longrightarrow \int_0^{\infty} \frac{d^d p}{(2\pi)^d} \left\{ \frac{1}{p^2+m^2} - \frac{a_1}{p^2+\Lambda_1^2} - \frac{a_2}{p^2+\Lambda_2^2} \right\}} \quad (18)$$

where $\boxed{a_1 + a_2 = 1}$ (19) to cancel the quadratic divergence, and an additional condition is imposed to make the coeff of $\frac{1}{p^4}$ vanish at large p

$$\boxed{-m^2 + \frac{a_1}{\Lambda_1^2} + \frac{a_2}{\Lambda_2^2} = 0} \quad (20)$$

Since $\Lambda_1^2, \Lambda_2^2 >> m^2$ this boils down to

$$\boxed{a_2 = -\frac{a_1 \Lambda_2^2}{\Lambda_1^2}} \quad (21)$$

One possible choice is $\boxed{\Lambda_2 = 2\Lambda_1}$ (22), $a_2 = -4a_1$

and $\boxed{a_1 = -\frac{1}{3} \qquad a_2 = \frac{4}{3}}$ (23)

Before tackling $I_2(\bar{k})$ let us do the sub-integral which will appear in Γ_4

$$\boxed{I_4(\bar{k}) = \int \frac{d^d p}{(2\pi)^d} \left\{ \frac{1}{(p^2+m^2)[(\bar{p}-\bar{k})^2+m^2)]} - m \to \Lambda \right\}} \quad (24)$$

$$= \frac{\Gamma(2)}{(\Gamma(1))^2}\int_0^1 d\alpha \int \frac{d^d p}{(2\pi)^d}\left\{\frac{1}{[(1-\alpha)(p^2+m^2)+\alpha\,((\bar{p}-\bar{k})^2+m^2)]^2} - m\to\lambda\right\} \quad (25)$$

Denominator $$D = (1-\alpha)(p^2+m^2)+\alpha[p^2+k^2-2\bar{k}\cdot\bar{p}+m^2] \quad (26)$$

$$= p^2 - 2\alpha\,\bar{k}\cdot\bar{p} + \alpha k^2 + m^2$$

Let $$\bar{p}' = \bar{p} - \alpha\bar{k} \quad (27)$$

$$p^2 - 2\alpha\,\bar{k}\cdot\bar{p} = \bar{p}'^2 - \alpha^2\bar{k}^2$$

$$D = \bar{p}'^2 + m^2 + k^2\alpha(1-\alpha) \quad (28)$$

$$\Rightarrow \quad I_4(\bar{k}) = \int_0^1 d\alpha \int_0^\infty \frac{d^d\bar{p}}{(2\pi)^d}\left\{\frac{1}{[\bar{p}'^2+m^2+k^2\alpha(1-\alpha)]^2} - m\to\lambda\right\} \quad (29)$$

shift the origin to $\bar{p}'$ so that there is no angular dependence

$$I_4(\bar{k}) = \frac{S_{d-1}}{(2\pi)^d}\int_0^1 d\alpha \int_0^\infty p'^{\,d-1}\,dp'\left\{\frac{1}{[p'^2+m^2+k^2\alpha(1-\alpha)]^2} - m\to\lambda\right\} \quad (30)$$

Let $$M^2 = m^2+k^2\alpha(1-\alpha) \quad (31)$$. define $$z = \frac{p'}{M} \quad (32)$$

$$\int_0^\infty \frac{p'^{\,d-1}\,dp'}{(p'^2+M^2)^2} = M^{d-4}\int_0^\infty \frac{dz\, z^{d-1}}{(z^2+1)^2} \quad (33)$$

converges in UV for $d<4$

Now we use Gradshteyn and Ryzhik 3·251(11)

$$\boxed{\int_0^\infty dx \, \frac{x^{\mu-1}}{(1+x^p)^\nu} = \frac{1}{p} B\left(\frac{\mu}{p}, \nu - \frac{\mu}{p}\right)} \quad (34)$$

where B is the Euler Beta function (not to be confused with the RG β-function!)

$$\boxed{B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}} \quad (35)$$

So
$$\int_0^\infty \frac{z^{d-1}\,dz}{(z^2+1)^2} = \frac{1}{2} B\left(\frac{d}{2}, 2-\frac{d}{2}\right) \quad (36)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)}$$

So
$$\boxed{I_4(\bar{k}) = \frac{S_{d-1}}{(2\pi)^d} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)}{2} \int_0^1 d\alpha \left\{ \frac{1}{\left[m^2+k^2\alpha(1-\alpha)\right]^{\frac{4-d}{2}}} - m \to \Lambda \right\}} \quad (37)$$

The key point to notice is that this integral is singular as $d \to 4$. Recall that the Γ function has poles at 0 and all the negative integers $-n$, with residue $\frac{(-1)^n}{n!}$

$\Rightarrow$ as $d \to 4$

$$\boxed{\Gamma\left(2-\frac{d}{2}\right) \to \frac{1}{2-\frac{d}{2}} = \frac{2}{4-d} = \frac{2}{\epsilon}} \quad (38)$$

The rest of the integral $\displaystyle\int_0^1 \frac{d\alpha}{[m^2+k^2\alpha(1-\alpha)]^{\frac{4-d}{2}}}$

is convergent as long as $m \neq 0$, or for $m=0$ $d>3$

At $m=0$, we can use the integral representation of the Euler Beta f^n

$$\boxed{B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}} \quad (39)$$

$$\int_0^1 d\alpha \frac{1}{[k^2\alpha(1-\alpha)]^{\frac{\epsilon}{2}}} = \frac{1}{(k^2)^{\frac{\epsilon}{2}}} \int_0^1 d\alpha\, \alpha^{-\frac{\epsilon}{2}} (1-\alpha)^{-\frac{\epsilon}{2}}$$

$$(40)$$

$$= \frac{1}{(k^2)^{\frac{\epsilon}{2}}} B\left(1-\frac{\epsilon}{2}, 1-\frac{\epsilon}{2}\right) = \boxed{\frac{1}{(k^2)^{\epsilon/2}} \frac{\left[\Gamma\left(1-\frac{\epsilon}{2}\right)\right]^2}{\Gamma(2-\epsilon)}}$$

Now we are ready to tackle $I_2(\bar{k})$. What we really want in order to calculate η is $I_2(\bar{k}) - I_2(0)$ for the special case $m^2=0$

$$\boxed{\tilde{I}_2(\bar{k}) = I_2(\bar{k}) - I_2(0) = \int \frac{d^dp_1\, d^dp_2}{(2\pi)^{2d}} \left[\frac{1}{p_1^2 p_2^2 (\bar{p}_1+\bar{p}_2-\bar{k})^2} - \frac{1}{p_1^2 p_2^2 (\bar{p}_1+\bar{p}_2)^2}\right]} \quad (41)$$

First do the $\bar{p}_2$ integral, which is the same as I_4

$$\boxed{\int \frac{d^d p_2}{(2\pi)^d} \frac{1}{p_2^2(\bar{p}_2+\bar{p}_1-\bar{k})^2} = \frac{S_{d-1}}{(2\pi)^d} \frac{\Gamma(\frac{d}{2})\Gamma(\frac{\epsilon}{2})}{2\Gamma(2-\epsilon)} \frac{[\Gamma(1-\frac{\epsilon}{2})]^2}{[(\bar{p}_1-\bar{k})^2]^{\epsilon/2}}} \quad (42)$$

Now use the Feynman trick again (14)

$$\boxed{\frac{1}{p_1^2[(\bar{p}_1-\bar{k})^2]^{\epsilon/2}} = \frac{\Gamma(1+\frac{\epsilon}{2})}{\Gamma(1)\,\Gamma(\frac{\epsilon}{2})} \int_0^1 d\alpha \frac{\alpha^{\frac{\epsilon}{2}-1}}{[\alpha(\bar{p}_1-\bar{k})^2+(1-\alpha)\,p_1^2]^{1+\frac{\epsilon}{2}}}} \quad (43)$$

Denominator $D = \alpha\,(p_1^2+k^2-2\bar{p}_1\cdot\bar{k}) + (1-\alpha)\,p_1^2$ (44)

$$= p_1^2 - 2\alpha\,\bar{p}_1\cdot\bar{k} + \alpha k^2 = \boxed{(\bar{p}_1-\alpha\bar{k})^2 + \alpha(1-\alpha)k^2 = D}$$

Now we impose a Pauli-Villars cutoff

$$\boxed{J = \int_0^1 d\alpha\,\alpha^{\frac{\epsilon}{2}-1}\int_0^\infty \frac{d^d p_1}{(2\pi)^d}\left\{\frac{1}{[(\bar{p}_1-\alpha\bar{k})^2+\alpha(1-\alpha)k^2]^{1+\frac{\epsilon}{2}}} - k\to 0 - \left[\frac{1}{[(\bar{p}_1-\alpha\bar{k})^2+\alpha(1-\alpha)k^2+\Lambda^2]} - k\to 0\right]\right\}} \quad (45)$$

With this cutoff there are no UV divergences, and no IR divergences either for small ϵ.

Shift $\bar{p}_1' = \bar{p}_1 - \alpha \bar{k}$

$$\boxed{J = \frac{S_{d-1}}{(2\pi)^d} \int_0^1 d\alpha \; \alpha^{\frac{\epsilon}{2}-1} \int_0^\infty p^{d-1}\, dp \left\{ \frac{1}{[p^2+\alpha(1-\alpha)k^2]^{1+\frac{\epsilon}{2}}} - \frac{1}{(p^2)^{1+\epsilon/2}} - \frac{1}{[p^2+\alpha(1-\alpha)k^2+\Lambda^2]^{1+\frac{\epsilon}{2}}} + \frac{1}{[p^2+\Lambda^2]^{1+\epsilon/2}} \right\}} \quad (46)$$

Consider

$$\int_0^\infty \left\{ \frac{p^{d-1}\, dp}{(p^2+a^2)^{1+\epsilon/2}} - \frac{p^{d-1}\, dp}{(p^2+a^2+\Lambda^2)^{1+\frac{\epsilon}{2}}} \right\}$$

$$= -\int_0^\infty p^{d-1}\, dp \int_{a^2}^{a^2+\Lambda^2} d\xi \; \underbrace{\frac{d}{d\xi} \frac{1}{(p^2+\xi)^{1+\epsilon/2}}}_{-(1+\epsilon/2)\frac{1}{(p^2+\xi)^{2+\epsilon/2}}} \qquad a^2 = \alpha(1-\alpha)k^2 \quad (47)$$

Interchange the order of integration

$$\int_0^\infty \frac{p^{d-1}\, dp}{(p^2+\xi)^{2+\epsilon/2}}$$ is convergent in the UV

Scale out ξ $\quad z = p/\sqrt{\xi}$

$$\xi^{\frac{d}{2}-2-\frac{\epsilon}{2}} \int_0^\infty \frac{z^{d-1}\, dz}{(z^2+1)^{4-d/2}} = \xi^{d-4}\, \frac{1}{2} B\left(\frac{d}{2}, 4-\frac{d}{2}-\frac{d}{2}\right)$$

$$= \frac{1}{2}\, \xi^{d-4}\, \frac{\Gamma(\frac{d}{2})\, \Gamma(4-d)}{\Gamma(4-\frac{d}{2})} \quad (48)$$

So $$\int_0^\infty p^{d-1}\,dp\left\{\frac{1}{(p^2+a^2)^{1+\frac{\epsilon}{2}}}-\frac{1}{(p^2+a^2+\Lambda^2)^{1+\frac{\epsilon}{2}}}\right\}$$

$$=\frac{1}{2}\frac{\Gamma(\frac{d}{2})\Gamma(4-d)}{\Gamma(4-\frac{d}{2})}\int_{a^2}^{\Lambda^2+a^2}\frac{d\xi}{\xi^{\epsilon}}\quad(1+\tfrac{\epsilon}{2})$$

$$\boxed{=\frac{1}{2}\frac{\Gamma(\frac{d}{2})\Gamma(\epsilon)}{\Gamma(4-\frac{d}{2})}\frac{1+\epsilon/2}{1-\epsilon}\left[(\Lambda^2+a^2)^{1-\epsilon}-(a^2)^{1-\epsilon}\right]}\qquad (49)$$

We still need to do the integral over α

$$\boxed{J=\frac{S_{d-1}}{(2\pi)^d}\frac{1}{2}\frac{\Gamma(\frac{d}{2})\Gamma(\epsilon)}{\Gamma(4-\frac{d}{2})}\frac{1+\epsilon/2}{1-\epsilon}\int_0^1 d\alpha\;\alpha^{\frac{\epsilon}{2}-1}\Big[\left[\Lambda^2+\alpha(1-\alpha)k^2\right]^{d-3}-(\alpha(1-\alpha))^{d-3}(k^2)^{d-3}\Big]}\qquad (50)$$

Since $k^2 \ll \Lambda^2$ we can expand

$$\boxed{\left[\Lambda^2+\alpha(1-\alpha)k^2\right]^{1-\epsilon}=(\Lambda^2)^{1-\epsilon}\left\{1+(1-\epsilon)\,\alpha(1-\alpha)\frac{k^2}{\Lambda^2}+\frac{(1-\epsilon)(\epsilon)}{2!}(\alpha(1-\alpha))^2\left(\frac{k^2}{\Lambda^2}\right)^2+\cdots\right\}}\qquad (51)$$

We can do the α integral term by term using

$$\boxed{B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}}\qquad (39)$$

$$J = \frac{S_{d-1}}{(2\pi)^d} \frac{1}{2} \frac{\Gamma(\frac{d}{2})\Gamma(\epsilon)}{\Gamma(4-\frac{d}{2})} \frac{1+\epsilon/2}{1-\epsilon} \quad \otimes$$

$$\left\{ -(k^2)^{1-\epsilon} \int_0^1 d\alpha \; \alpha^{-\frac{\epsilon}{2}} (1-\alpha)^{1-\epsilon} + (\Lambda^2)^{1-\epsilon} \int_0^1 d\alpha \; \alpha^{\frac{\epsilon}{2}-1} \right.$$

$$\left. + (1-\epsilon) \frac{k^2}{\Lambda^2} \int_0^1 d\alpha \; \alpha^{\frac{\epsilon}{2}} (1-\alpha) + \cdots \right\}$$

$$\boxed{J = \frac{S_{d-1}}{(2\pi)^d} \frac{1}{2} \frac{\Gamma(\frac{d}{2})\Gamma(\epsilon)}{\Gamma(2+\frac{\epsilon}{2})} \frac{1+\epsilon/2}{1-\epsilon} \left\{ -(k^2)^{1-\epsilon} \frac{\Gamma(1-\frac{\epsilon}{2})\Gamma(2-\epsilon)}{\Gamma(3-\frac{3}{2}\epsilon)} + (\Lambda^2)^{1-\epsilon} \frac{2}{\epsilon} + (1-\epsilon)\frac{k^2}{\Lambda^2} \frac{\Gamma(1+\frac{\epsilon}{2})\Gamma(2)}{\Gamma(3+\frac{\epsilon}{2})} + \cdots \right\}} \quad S2$$

We are interested in the k^2 dependent terms only and we know that $\frac{k}{\Lambda} \to 0$

Also recall

$$\Gamma(1+x) = x\,\Gamma(x)$$

$$\Rightarrow \quad \Gamma(\epsilon) = \Gamma(-1+\epsilon)(-1+\epsilon)$$

$$\Rightarrow \quad \frac{\Gamma(\epsilon)}{1-\epsilon} = -\Gamma(-1+\epsilon)$$

$$\Gamma(2+\epsilon/2) = (1+\epsilon/2)\,\Gamma(1+\epsilon/2)$$

$$\boxed{J = + \frac{S_{d-1}}{(2\pi)^d} \frac{1}{2} \frac{\Gamma(\frac{d}{2})\Gamma(-1+\epsilon)}{\Gamma(1+\frac{\epsilon}{2})} (k^2)^{1-\epsilon} \frac{\Gamma(1-\frac{\epsilon}{2})\Gamma(2-\epsilon)}{\Gamma(3-\frac{3}{2}\epsilon)}} \quad S3$$

Now go back to the integral

$$\int_0^\infty \frac{p^{d-1}\, dp}{\left[p^2+\alpha(1-\alpha)k^2\right]^{1+\frac{\epsilon}{2}}}$$

Suppose we work in a range of d where the integral is convergent.

$$1+\frac{\epsilon}{2} > 2-\frac{\epsilon}{2} \quad \text{or} \quad \epsilon > 1$$

Then, using

$$\boxed{\int_0^\infty dx\, \frac{x^{\mu-1}}{(1+x^p)^\nu} = \frac{1}{p}\, B\left(\frac{\mu}{p}, \nu-\frac{\mu}{p}\right)} \quad (34)$$

we get

$$\int_0^\infty dp\, \frac{p^{d-1}}{\left[p^2+\alpha(1-\alpha)k^2\right]^{1+\frac{\epsilon}{2}}} = (\alpha(1-\alpha))^{\frac{d}{2}-1-\frac{\epsilon}{2}} (k^2)^{\frac{d}{2}-1-\frac{\epsilon}{2}}$$

$$\frac{1}{2}\, B\left(\frac{d}{2}, 1+\frac{\epsilon}{2}-\frac{d}{2}\right)$$

$$\frac{d}{2} = 2-\frac{\epsilon}{2}$$

$$= \left[\alpha(1-\alpha)\right]^{1-\epsilon} (k^2)^{1-\epsilon}\, \frac{1}{2}\, \frac{\Gamma(\frac{d}{2})\Gamma(-1+\epsilon)}{\Gamma(1+\frac{\epsilon}{2})}$$

Integrating over α after multiplying by $\alpha^{\frac{\epsilon}{2}-1}$

$$\int_0^1 d\alpha \; \alpha^{-\frac{\epsilon}{2}} \; (1-\alpha)^{1-\epsilon} = \frac{\Gamma(1-\epsilon/2)\,\Gamma(2-\epsilon)}{\Gamma(3-\frac{3}{2}\epsilon)}$$

So in this range of ϵ

$$\boxed{J = \frac{S_{d-1}}{(2\pi)^d} (k^2)^{1-\epsilon} \frac{1}{2} \frac{\Gamma(\frac{d}{2})\,\Gamma(-1+\epsilon)}{\Gamma(1+\frac{\epsilon}{2})} \frac{\Gamma(1-\frac{\epsilon}{2})\,\Gamma(2-\epsilon)}{\Gamma(3-\frac{3}{2}\epsilon)}} \quad (54)$$

Identical to (53) !!

This is the basis of dimensional regularization in which one computes integrals without worrying about convergence. More precisely, one computes integrals in a range of d in which they converge, and then analytically continues to the d one is working in.

From (42), (43), (45), (54)

$$\tilde{I}_2(\bar{k}) = \frac{S_{d-1}^2}{(2\pi)^{2d}} \frac{\Gamma(\frac{d}{2})\cancel{\Gamma(\frac{\epsilon}{2})}}{2\cancel{\Gamma(2-\epsilon)}} \left[\Gamma(1-\tfrac{\epsilon}{2})\right]^2 \frac{\cancel{\Gamma(1+\frac{\epsilon}{2})}}{\Gamma(1)\,\cancel{\Gamma(\frac{\epsilon}{2})}}$$

$$\times \; (k^2)^{1-\epsilon} \frac{1}{2} \frac{\Gamma(\frac{d}{2})\,\Gamma(-1+\epsilon)}{\cancel{\Gamma(1+\frac{\epsilon}{2})}} \frac{\Gamma(1-\frac{\epsilon}{2})\,\cancel{\Gamma(2-\epsilon)}}{\Gamma(3-\frac{3}{2}\epsilon)}$$

$$\boxed{\tilde{I}_2(\bar{k}) = \left[\frac{S_{d-1}}{(2\pi)^d}\right]^2 \frac{\Gamma(\frac{d}{2})\left[\Gamma(1-\frac{\epsilon}{2})\right]^3 \Gamma(-1+\epsilon)}{4\,\Gamma(3-\frac{3}{2}\epsilon)} (k^2)^{1-\epsilon}} \quad (55)$$