Angular Impulse and Collisions

We have learnt that in analogy with Newton's II Law for CM motion

$$\boxed{\vec{F}_{tot} = M\vec{a}_{CM} = \frac{d}{dt}\vec{P}_{CM}} \quad (1)$$

we have the angular eqⁿ

$$\boxed{\tau_{tot} = I\alpha = \frac{d}{dt}(I\omega) = \frac{dL}{dt}} \quad (2)$$

for a fixed axis rotation.

Recall that in talking about collisions we found it useful to define the impulse $\vec{\mathcal{J}}$

$$\boxed{\vec{\mathcal{J}} = \int_{t_i}^{t_f} \vec{F}\, dt} \quad (3)$$

I will use $\vec{\mathcal{J}}$ for impulse and I for moment of inertia. The impulse-momentum thm says

$$\boxed{\vec{\mathcal{J}} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i} \quad (4)$$

There are many cases where it is useful to define the angular analogue of impulse

$$\boxed{\mathcal{J}_{ang} = \int_{t_i}^{t_f} \tau \, dt}$$ (5) (fixed axis)

and the angular version of the impulse momentum theorem

$$\boxed{\mathcal{J}_{ang,tot} = \Delta L = L_f - L_i}$$ (6)

To put this on a firm foundation we need to define torque and angular momentum vectorially for an arbitrary collection of point masses.

For a single point mass $\boxed{M_\beta}$ (7) at position $\boxed{\vec{r}_\beta}$ (8) travelling at velocity $\boxed{\vec{v}_\beta}$ (9)

$$\boxed{\vec{L}_\beta = \vec{r}_\beta \times \vec{p}_\beta = M_\beta \, \vec{r}_\beta \times \vec{v}_\beta}$$ (10)

$$\boxed{\vec{L}_{tot} = \sum_\beta \vec{L}_\beta = \sum_\beta M_\beta \, \vec{r}_\beta \times \vec{v}_\beta}$$ (11)

Suppose we now specialize to the case when the masses are part of a rigid body, with a CM velocity $\boxed{\vec{v}_{CM}}$ (12), a fixed axis of rotation $\boxed{\hat{n}}$ (13) going through the CM, and an angular velocity $\boxed{\omega}$ (14) around the axis of rotation.

$$\boxed{\vec{v}_{rot}(\vec{r}) = \omega \, \hat{n} \times (\vec{r} - \vec{R}_{CM})}$$ (15)

where as usual

$$\boxed{M \vec{R}_{CM} = \sum_\beta M_\beta \bar{r}_\beta} \quad (16)$$

So

$$\boxed{\vec{v}(\bar{r}) = \vec{v}_{CM} + \vec{v}_{rot}(\bar{r}) = \vec{v}_{CM} + \omega \hat{n} \times (\bar{r} - \vec{R}_{CM})} \quad (17)$$

$$\boxed{\vec{L}_{tot} = \sum_\beta M_\beta \bar{r}_\beta \times \left(\vec{v}_{CM} + \omega \hat{n} \times (\bar{r}_\beta - \vec{R}_{CM}) \right)} \quad (18)$$

$$\boxed{= \left(\sum_\beta M_\beta \bar{r}_\beta \right) \times \vec{v}_{CM} + \omega \sum_\beta M_\beta \bar{r}_\beta \times \left[\hat{n} \times (\bar{r}_\beta - \vec{R}_{CM}) \right]} \quad (19)$$

Now $\sum_\beta M_\beta \bar{r}_\beta = M \bar{R}_{CM}$

So the 1st term is just the angular momentum of a point mass M located at $\vec{R}_{CM}$, moving at $\bar{v}_{CM}$. Let us call this

$$\boxed{\begin{aligned} \vec{L}_{trans} &= \text{translational angular momentum} \\ &= M \bar{R}_{CM} \times \bar{v}_{CM} \end{aligned}} \quad (20)$$

To make further progress define the position relative to the CM

$$\boxed{\vec{\tilde{r}}_\beta = \bar{r}_\beta - \bar{R}_{CM}.} \quad (21)$$

$$\boxed{\sum_\beta M_\beta \underset{\sim}{\vec{r}}_\beta = \sum_\beta M_\beta \vec{r}_\beta - M\vec{R}_{CM} = 0} \quad (22)$$

Now consider the second term in $\vec{L}_{tot}$

This term clearly has to do with the rigid rotation of the body around its CM, because of the ω in front.

(23)

we will call it $\boxed{\vec{L}_{rot} = \text{Rotational } \vec{L}}$

$$\boxed{\begin{aligned}\vec{L}_{rot} &= \omega \sum_\beta M_\beta \left(\underset{\sim}{\vec{r}}_\beta + \vec{R}_{CM}\right) \times \left[\hat{n} \times \underset{\sim}{\vec{r}}_\beta\right] \\ &= \omega \sum_\beta M_\beta \underset{\sim}{\vec{r}}_\beta \times \left[\hat{n} \times \underset{\sim}{\vec{r}}_\beta\right] + \omega \sum_\beta M_\beta \vec{R}_{CM} \times \left[\hat{n} \times \underset{\sim}{\vec{r}}_\beta\right]\end{aligned}} \quad (24)$$

The last term vanishes. To see this note that $\vec{R}_{CM}$ and $\hat{n}$ are independent of β

So $\boxed{\text{last term} = \vec{R}_{CM} \times \left[\hat{n} \times \sum_\beta M_\beta \underset{\sim}{\vec{r}}_\beta\right] = 0} \quad (25)$

So $\boxed{\vec{L}_{rot} = \sum_\beta M_\beta \underset{\sim}{\vec{r}}_\beta \times \left[\hat{n} \times \underset{\sim}{\vec{r}}_\beta\right]} \quad (26)$

This looks complicated but is actually simple. Since the axis is fixed we want $\vec{L}_{rot}$ along the axis of rotation

$$\vec{L}_{rot} \cdot \hat{n} = \omega \sum_{\beta} M_\beta \; \hat{n} \cdot \left[\bar{\underset{\sim}{r}}_\beta \times (\hat{n} \times \bar{\underset{\sim}{r}}_\beta) \right]$$

$$= \omega \sum_{\beta} M_\beta \; (\hat{n} \times \bar{\underset{\sim}{r}}_\beta) \cdot (\hat{n} \times \bar{\underset{\sim}{r}}_\beta)$$

$$= \omega \sum_{\beta} M_\beta \left(r_\beta \sin \theta_\beta \right)^2 = \omega \sum_{\beta} M_\beta \left(\delta(\bar{r}_\beta) \right)^2 \quad (27)$$

$$\boxed{\delta(\bar{r}_\beta) = \text{distance from axis of rotation}} \quad (28)$$

So $\boxed{\vec{L}_{rot} \cdot \hat{n} = I_{CM}\, \omega}$ (29)

Put it together. For a fixed axis rotating body

$$\boxed{\vec{L}_{tot} = M \vec{R}_{CM} \times \bar{v}_{CM} + \hat{n}\, I_{CM}\, \omega} \quad (30)$$

Now let's derive the angular analogue of Newton's II law. For a point particle

$$\boxed{\vec{L}_\beta = M_\beta \, \bar{r}_\beta \times \bar{v}_\beta} \quad (31)$$

$$\boxed{\frac{d\bar{L}_\beta}{dt} = M_\beta \underbrace{\frac{d\bar{r}_\beta}{dt}}_{\vec{v}_\beta} \times \bar{v}_\beta + M_\beta \, \bar{r}_\beta \times \frac{d\bar{v}_\beta}{dt}} \quad (32)$$

Since $\bar{v}_\beta \times \bar{v}_\beta = 0$ and $M_\beta \dfrac{d\bar{v}_\beta}{dt} = \bar{F}_{tot,\beta}$

$$\boxed{\frac{d\bar{L}_\beta}{dt} = \bar{r}_\beta \times \bar{F}_{tot,\beta} = \vec{\tau}_\beta} \quad (33)$$

This is the vector definition of torque.

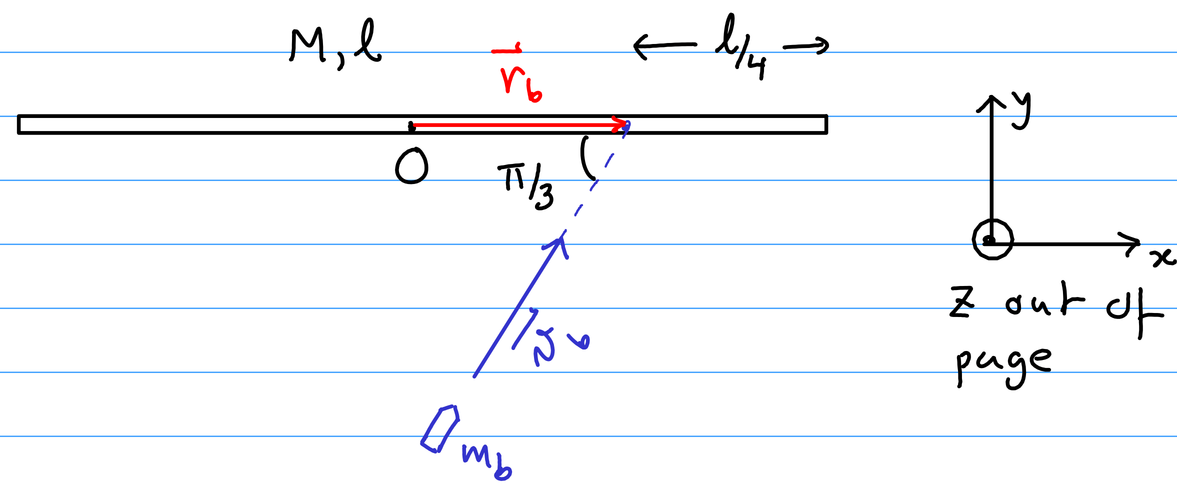
Now consider a rigid body. The torques can be internal (between the point masses making up the body) or external.

As usual, by Newton's III law, under reasonable assumptions all internal torques cancel. So

$$\boxed{\frac{d\vec{L}_{tot}}{dt} = \vec{\tau}_{ext} = \sum_\beta \vec{r}_\beta \times \vec{F}_{\beta,ext}} \quad (34)$$

Now we are ready to consider collisions where angular momentum is as important as momentum.

Example 1: A long thin bar of mass $\boxed{M = 9.95\ \text{kg}}$ (35) and length $\boxed{\ell = 2\text{m}}$ (36) sits on a frictionless horizontal surface. A bullet of mass $\boxed{m_b = 0.05\ \text{kg}}$ (37) and velocity $\boxed{400\ \text{m/s}}$ (38) strikes the bar as shown and embeds itself.



The key is the conservation of momentum and angular momentum.

Choose the final CM of the bar+bullet just after the collision as the origin O. Since $m_b \ll M$ this is the same as the CM of the bar to an excellent approximation

With respect to O the total initial momentum is

$$= \boxed{\vec{P}_{i,tot} = m_b \vec{v}_b = 20 \, kg\, m/s \left(\frac{\hat{i}}{2} + \frac{\sqrt{3}}{2} \hat{j} \right)} \quad (39)$$

The initial angular momentum is

$$\boxed{\vec{L}_{i,tot} = m_b \vec{r}_b \times \vec{v}_b} \quad (40)$$

The $\vec{r}_b$ can be chosen as the point where the bullet hits the bar.

$$\boxed{\vec{r}_b = \frac{l}{4} \hat{i}} \quad (41)$$

$$\Rightarrow \boxed{\begin{aligned} m_b \vec{r}_b \times \vec{v}_b &= \frac{l}{4} \times 20 \, kg\, m/s \left(\hat{i} \times \frac{\hat{i}}{2} + \hat{i} \times \frac{\sqrt{3}}{2} \hat{j} \right) \\ &= \frac{2 \times 20}{4} \frac{\sqrt{3}}{2} \hat{k} = 5\sqrt{3} \, kg \frac{m^2}{s} \hat{k} \end{aligned}} \quad (42)$$

In the final state the system of bar has a CM velocity $\vec{v}_{CM,f}$ and an angular velocity around the CM of ω_f

$$\boxed{\vec{P}_{f,tot} = (M+m_b)\,\vec{v}_{CM,f}} \quad (43)$$

$\Rightarrow$ Conservation of momentum implies

$$\boxed{\vec{P}_{i,tot} = 20\ \text{kg m/s}\left(\frac{\hat{i}}{2} + \frac{\sqrt{3}}{2}\hat{j}\right) = 10\ \text{kg}\ \vec{v}_{CM,f}} \quad (44)$$

$$\boxed{\vec{v}_{CM,f} = 2\ \text{m/s}\left(\frac{\hat{i}}{2} + \frac{\sqrt{3}}{2}\hat{j}\right)} \quad (45)$$

Final angular momentum about CM is

$$\boxed{I_{CM}\,\omega_f = \frac{1}{12} M\ell^2 \omega_f = \frac{10}{12} \cdot 4\,\omega_f = \frac{10}{3}\omega_f} \quad (46)$$

So $\boxed{\underbrace{5\sqrt{3}}_{L_i} = \underbrace{\frac{10}{3}\omega_f}_{L_f}}$ (47) $\Rightarrow$ $\boxed{\omega_f = \frac{3\sqrt{3}}{2}\ \frac{\text{rad}}{\text{s}}}$ (48)

Let us see how much KE has been converted to heat

$$\boxed{K_i = \frac{1}{2}(0.05)\,16\times10^4 = 4000\ \text{J}} \quad (49)$$

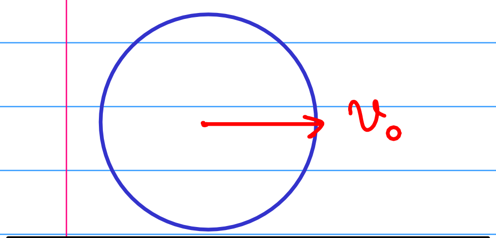
$$K_f = \frac{1}{2} M v_{CM,f}^2 + \frac{1}{2} I \omega_f^2$$

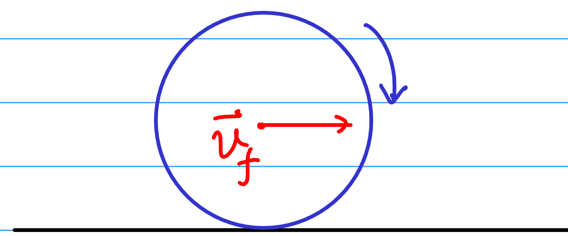
$$= \frac{1}{2}\times 10 \times 4 + \frac{1}{2}\,\frac{10}{3}\left(\frac{3\sqrt{3}}{2}\right)^2 =$$

$$20\ \text{J} + 11.25\ \text{J} = 31.25\ \text{J}$$

energy converted into heat = 3968.75 J

<u>Example 2</u>: A ball (sphere of uniform density) of mass M and radius R is given a horizontal CM velocity of v_0 just above a horizontal surface. Initially the ball is not rotating around its CM. It hits the floor and skids until it starts rolling w/o slipping. What is its CM velocity v_f when this occurs?



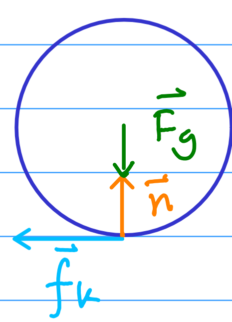


$$\boxed{\omega_i = 0} \quad (50)$$

$$\boxed{\omega_f = -\frac{v_f}{R}} \quad (51)$$

Since it skids $W_{nc} \neq 0$ and Mechanical energy is not conserved. The way to solve the problem is to realize that the force of friction $\vec{f}_k$ gives the ball a linear impulse and an angular impulse.

While the ball skids, its FBD is



$a_y = 0$

$$\Rightarrow \boxed{n = Mg} \quad (52)$$

Suppose it skids for a time Δt. Impulse

$$\boxed{\vec{\mathcal{J}}_{tot} = -\hat{i}\, f_k\, \Delta t} \quad (53)$$

$$\boxed{\mathcal{J}_{ang,tot} = -f_k R\, \Delta t} \quad (54)$$

Using both versions of the impulse-momentum theorem

$$\boxed{-\hat{i}\, f_k\, \Delta t = \Delta \vec{p}_{CM} = M(v_f - v_0)\,\hat{i}} \quad (55) \text{ Linear}$$

and

$$\boxed{-f_k R\, \Delta t = \Delta L = I\omega_f - 0} \quad (56) \text{ Angular}$$

When it rolls w/o slipping $\boxed{\omega_f = \frac{-v_f}{R}}$ (57)

So $$f_k R\, \Delta t = \frac{I v_f}{R}$$

$$\boxed{f_k\, \Delta t = \frac{I v_f}{R^2}} \quad (58)$$

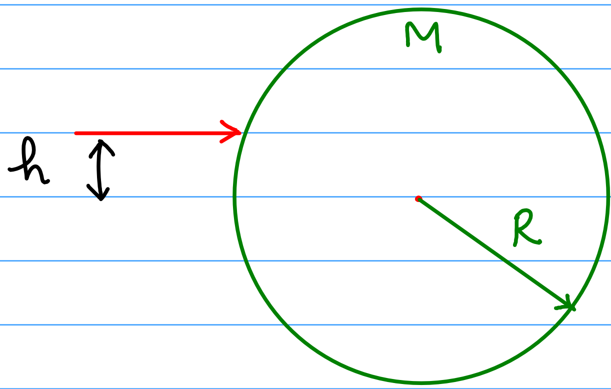
Plug into linear impulse-momentum thm

$$\boxed{-\frac{I v_f}{R^2} = M(v_f - v_0)} \quad (59)$$

$$M v_0 = v_f\left(M + \frac{I}{R^2}\right) = M v_f\left(1 + \frac{2}{5}\right)$$

$$\boxed{v_f = \frac{5}{7} v_0} \quad (60)$$

Example 3: Suppose we want to hit a pool ball with a cue in the horizontal direction in such a way that it never skids but rolls w/o slipping from the beginning. At what height should the cue contact the ball?



Suppose the impulse $\mathcal{J}$ is given at a height h above the CM

$$\boxed{\vec{\mathcal{J}} = \Delta \vec{p} = \vec{p}_f - \vec{p}_i = \mathcal{J}\,\hat{i}} \quad (61)$$

since $\boxed{\vec{v}_i = \omega_i = 0}$ (62)

$$\boxed{p_f = M v_{CM,f} = \mathcal{J}} \quad (63)$$

For the angular part (64)

$$\boxed{\mathcal{J}_{ang} = -\mathcal{J}\,h = \Delta L = I\omega_f - I\omega_i = I\,\omega_f}$$

We want rolling without slipping

$$\Rightarrow \quad \omega_f = -\frac{v_{CM,f}}{R}$$

or $$\boxed{\mathcal{J} h = \frac{2}{5} M R^2 \frac{v_{CM,f}}{R}} \quad (65)$$

and $$\boxed{\mathcal{J} = M v_{CM,f}} \quad (66)$$

Take the ratio

$$\boxed{h = \frac{2}{5} R} \quad (67)$$